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HOW DO PROSPECTIVE MATHEMATICS TEACHERS APPROACH PROOF AND REFUTATION? A FOCUS ON ABDUCTIVE REASONING

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ABSTRACT This study aims to explore the abductive reasoning strategies used by prospective mathematics teachers in proving and refuting mathematical statements. This study used a qualitative method with a case study design. Fourteen third-year prospective mathematics teachers were involved in this study and were then grouped according to the characteristics of the type of abductive reasoning they used. Data collection techniques included tests given to all prospective mathematics teachers and interviews conducted with five prospective mathematics teachers selected based on their type of abductive reasoning. The data obtained was analyzed through stages that included data condensation, data display, and conclusion drawing. Technique triangulation was used to check the validity of the research findings. In general, it was found that the types of abductive reasoning strategies used by prospective mathematics teachers in proving included fact optimization and mistaken fact. Meanwhile, in refuting mathematical statements, there are three types of abductive reasoning used by students, consisting of fact optimization, mistaken fact, and factual error. The results of this study provide insight into how abductive reasoning contributes to formulating mathematical conjectures and can help educators design relevant learning strategies to support the improvement of students' proof and refutation abilities.

Keywords: abductive reasoning, proof, refufation

ABSTRAK Penelitian ini bertujuan untuk mengeksplorasi strategi penalaran abduktif yang digunakan oleh calon guru matematika dalam membuktikan dan menyangkal pernyataan matematika. Penelitian ini menggunakan metode kualitatif dengan desain studi kasus. Sebanyak 14 calon guru matematika tingkat tiga terlibat dalam penelitian ini dan kemudian dikelompokkan berdasarkan karakteristik jenis penalaran abduktif yang mereka gunakan. Teknik pengumpulan data meliputi tes yang diberikan kepada seluruh calon guru matematika dan wawancara yang dilakukan dengan lima calon guru matematika yang dipilih berdasarkan jenis penalaran abduktif mereka. Data yang diperoleh dianalisis melalui tahapan kondensasi data, penyajian data, dan penarikan kesimpulan. Triangulasi teknik digunakan untuk memeriksa keabsahan temuan penelitian. Secara umum, ditemukan bahwa jenis strategi penalaran abduktif yang digunakan oleh calon guru matematika dalam pembuktian meliputi *fact optimization* dan *mistaken fact*. Sementara itu, dalam menyangkal pernyataan matematika, terdapat tiga jenis penalaran abduktif yang digunakan oleh mahasiswa, yaitu

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fact optimization, *mistaken fact*, dan *factual error*. Hasil penelitian ini memberikan wawasan tentang bagaimana penalaran abduktif berkontribusi dalam merumuskan konjektur matematika dan dapat membantu pendidik merancang strategi pembelajaran yang relevan untuk mendukung peningkatan kemampuan pembuktian dan penyangkalan matematika mahasiswa.

Kata-kata kunci: penalaran abduktif, pembuktian, penyangkalan.

INTRODUCTION

The ability to construct proofs holds a central role in mathematics learning (Gabriel et al., 2020; Hein & Prediger, 2024; Wasserman et al., 2018). Proofs not only establish the validity of mathematical solutions (Hamami & Morris, 2020) but also enhance students' understanding and support the development of mathematical knowledge (Zengin, 2017). The proving process is often accompanied by refutation, where students identify weaknesses, confront counterexamples, and refine conjectures, thus engaging with the dynamic nature of mathematical knowledge (Komatsu & Jones, 2022). Proof and refutation refer to the activity of proving a conjecture, confronting counterexamples that invalidate the conjecture or proof, and subsequently refining the proof by addressing those counterexamples (Komatsu, 2016). Refutation serves to improve conjectures by compelling reasoners to determine the broadest domain in which their claims hold (Creager, 2022). Lakatos, in Proofs and Refutations, describes mathematics as a dynamic process where conjectures, proofs, and refutations interact continuously (Alcock & Attridge, 2023).

Although proof and refutation are essential components of mathematics, prospective mathematics teachers still demonstrate weak abilities in both aspects (Güler, 2016; Putra et al., 2023),. These weaknesses include difficulties in connecting premises to intended conclusions (Budiarto & Artiono, 2019; Miyazaki et al., 2017), challenges in initiating proofs (Badjeber et al., 2025; Fu et al., 2022; Siswono et al., 2020),, limited ability to formulate relevant conjectures or assumptions in the proving process (Aisyah et al., 2023) and difficulties in understanding the roles of examples and counterexamples (Bergwall, 2021). Pre-service teachers also struggle with constructing refutations and tend to rely more frequently on local counterexamples (Creager, 2022). Moreover, they may fail to identify local counterexamples when validating claimed proofs, leading to an inability to recognize weaknesses in arguments (Ko & Knuth, 2013). Coordinating all conditions in a problem to generate valid counterexamples thus poses a significant challenge (Komatsu, 2017).

When constructing proofs, students are required to reason and argue by writing their solutions according to the steps they know (Ramandani et al., 2024), making reasoning an inseparable part of proof construction. One of the purposes of reasoning is to establish coherent causal relationships between data and claims.

Through mathematical reasoning, students can formulate conjectures, construct proofs, manipulate mathematical problems, and draw accurate and valid conclusions (Sari et al., 2022).

Hypothesis-based empirical investigation is often interpreted as deductive reasoning, whereas prospective mathematics teachers also seek explanations for phenomena, which involves another form of reasoning—abductive reasoning. Abductive reasoning differs from deduction and induction, yet it plays an equally important role in acquiring new knowledge (Bellucci, 2018). Peirce emphasized that abductive reasoning functions to formulate plausible conjectures and then explore them to reach conclusions whose validity can be justified (Hoffmann, 2018; Reid, 2018). Abductive reasoning has been adopted to foster discovery and enhance students' creativity (Ferguson, 2019) as well as to support students' modeling activities in mathematics learning (Park & Lee, 2016). It also contributes to generating new ideas (O'Reilly, 2016), making claims about the validity of questions (Wu et al., 2016) and formulating conjectures in geometry (Baccaglini-Frank, 2019).

Umberto Eco proposed three types of abductive reasoning: overcoded, undercoded, and creative abduction (Pedemonte & Reid, 2011). These types are classified based on the number and form of rules that can be selected during problem-solving. Furthermore, Hidayah et. al.,(2020) categorized abductive reasoning in problem solving into four categories: creative conjectures, fact optimization, factual error, and mistaken fact. Such forms of abduction may pose cognitive challenges both during argumentation and in constructing proofs (Reid, 2018). On the other hand, abduction helps students formulate claims that can later be transformed into deductive reasoning within the proving process (Kaplan et al., 2021). Students may also encounter counterexamples that invalidate their proofs and then engage in creative abduction, through which specific theorems are generated to address those counterexamples (Komatsu & Jones, 2022). Abductive strategies require the ability to articulate and apply essential concepts relevant to the given premises in order to establish intermediate targets before concluding (Salsabila et al., 2021; Shodikin, 2017).

Previous studies in mathematics education over the past decade have examined aspects involving proof, refutation, and abductive reasoning. Several studies have explored the role of abductive reasoning in conjecturing and proving (Baccaglini-Frank, 2019; Kaplan et al., 2021; Pedemonte, 2018), investigated the use of abductive reasoning in mathematical proof construction (Widadah et al., 2024), making an argument for indirect proof (Antonini, 2019) and examined the use of local counterexamples in refutation (Komatsu & Jones, 2022). Other studies have explored the types of abductive reasoning employed by prospective mathematics teachers in problem solving (Hidayah et al., 2020), as well as analyzed students' thinking processes within each type of abductive reasoning and their impact on the formation of new schemas (Shodikin et al., 2021). Research on proof strategies has primarily focused on deductive and inductive processes, such as identifying proof strategies used by students,(Miyazaki et al., 2024; Siswono et al., 2020),), examining students' weaknesses in proof construction (Ndemo, 2019), investigating the validation and modification of proofs (Komatsu et al., 2017), and exploring the interpretation, understanding, and use of generic examples as part of pre-service teachers' proving and justification activities (Dogan & Williams-Pierce, 2021). Furthermore, other studies have examined the role of counterexamples (Roh & Lee, 2017), discussed students' understanding of refutation and their performance in abstract conditional inference (Alcock & Attridge, 2023), investigated pre-service teachers' geometric refutations (Creager, 2022), explored the use of refutation as feedback to produced proofs (Pinto & Cooper, 2022), and identified the implications of refutation in students' arguments (Cervantes-Barraza & Cabañas Sánchez, 2022).

Previous studies have primarily examined students' proof performance deductively, focusing on the role of abductive reasoning in conjecturing, proving, and refuting through local counterexamples. However, little attention has been given to the specific abductive strategies used by prospective mathematics teachers in proof and refutatio. Based on this gap, we conducted a study aimed at exploring the abductive reasoning strategies used by prospective mathematics teachers in proving and refuting mathematical statements. We adopted the framework of Hidayah et al. to map the variations of abductive strategies employed in the processes of proof and refutation, including the exploration of local or global counterexamples to strengthen mathematical arguments. The findings of this study may provide insights into how abductive reasoning contributes to formulating mathematical conjectures and addressing counterexamples. This knowledge can enrich the literature on mathematical thinking strategies and support educators in designing instructional strategies to foster prospective mathematics teachers' proof abilities.

METHODS

This study employed a qualitative case study design to explore prospective mathematics teachers' abductive reasoning in proof and refutation. Fourteen third-year students at UIN Datokarama Palu were purposively selected due to their prior experience with proof-related courses. Data were collected using a proof-based diagnostic test and semi-structured interviews. The validated test consisted of two open-ended items: one requiring students to construct a formal proof and another to refute an incorrect statement using counterexamples or logical arguments. The research procedure involved administering the test, analyzing students' written responses, categorizing them based on abductive reasoning indicators, and selecting one representative participant from each category for follow-up interviews. Data from the test were analyzed using qualitative content analysis guided by indicators of abductive reasoning, such as hypothesis generation, plausibility judgment, and explanatory inference, resulting in two categories in proof construction and three in refutation. To enhance credibility, triangulation was conducted by comparing test and interview data. The interview guide was semi-structured and specifically designed to explore students' abductive strategies, focusing on how they generated hypotheses, constructed plausible explanations, selected and revised ideas, and justified conclusions during the processes of proof and refutation, with follow-up questions used to obtain deeper insights into their reasoning.

The data from tests and interviews were analyzed through data condensation, data display, and conclusion drawing (Miles et al., 2018). Prospective mathematics teachers responses were categorized into abductive reasoning types (Fact Optimization, Mistaken Fact, Factual Error, Creative Conjecture) to simplify complex information. The condensed data were then organized in distribution tables and supported with narrative accounts of students' reasoning in proof and refutation. Conclusions were drawn by identifying the characteristics of each reasoning strategy. Data validity was ensured through methodological triangulation by comparing test results with interview data. The descriptions of each abductive reasoning strategy are as follows.

Tabel 1. Description of Abductive Reasoning Characteristics (Hidayah et al., 2020)

Types of Abductive Reasoning	Characteristic Description
Creative conjectures	Prospective mathematics teachers generate new ideas to solve the given problem
Fact optimization	Prospective mathematics teachers propose a conjectured answer and then confirm it through deductive reasoning.
Factual error	Prospective mathematics teachers use incorrect facts outside the problem to attempt a solution
Mistaken fact	Prospective mathematics teachers assume information given in the problem as a known fact

FINDING AND DISCUSSION

Based on the test results, the types of abductive reasoning strategies used by prospective mathematics teachers in proof construction were identified as fact optimization and mistaken fact. Ten students demonstrated fact optimization reasoning, while four students exhibited mistaken fact reasoning. Furthermore, in refuting mathematical statements, three types of abductive reasoning were identified: nine students employed fact optimization, four students used mistaken fact reasoning, and one student demonstrated a factual error. A more detailed distribution of these findings is presented in the following table 2.

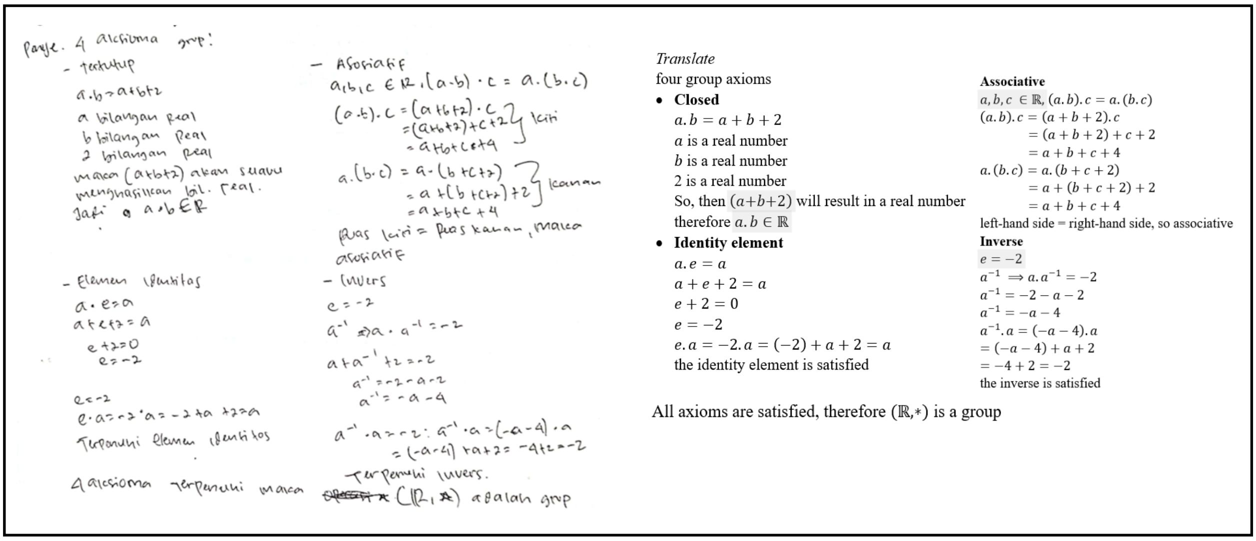
Tabel 2. Detailed Types of Prospective Theacers' Abductive Reasoning

Types of Abductive Reasoning	Proof	Refutation
Creative Conjectures	0	0
Fact Optimization	10	9
Factual Error	0	1
Mistaken Fact	4	4

Prospective Mathematics Teachers' Abductive Reasoning Strategies in Proving Mathematical Statements

1. Abductive Reasoning Fact Optimization by S6

S6 was able to solve the problem accurately by utilizing all the given facts in the task, namely that $a * b = a + b + 2$, and by clearly identifying what was being asked. In the problem-solving process, S6 also employed relevant external knowledge, specifically that in order to demonstrate a set is a group, it must satisfy four properties: closure under its binary operation, associativity, the existence of an identity element, and the existence of inverses. S6's written response is presented in Figure 1 below.



Pasye. 4 aksioma grup!
- tertutup
  $a.b = a+b+2$
  a bilangan real
  b bilangan real
  2 bilangan real
  maka $(a+b+2)$ akan selalu menghasilkan bil. real.
  Jadi $a.b \in \mathbb{R}$
- Elemen identitas
  $a.e = a$
  $a+e+2 = a$
  $e+2 = 0$
  $e = -2$
  $e = -2$
  $e.a = -2.a = -2+a+2 = a$
  Terpenuhi elemen identitas
- Asosiatif
  $a,b,c \in \mathbb{R}, (a.b).c = a.(b.c)$
  $(a.b).c = (a+b+2).c = (a+b+2)+c+2 = a+b+c+4$ kiri
  $a.(b.c) = a.(b+c+2) = a+(b+c+2)+2 = a+b+c+4$ kanan
  Ruas kiri = Ruas kanan, maka asosiatif
- Invers
  $e = -2$
  $a^{-1} \Rightarrow a.a^{-1} = -2$
  $a+a^{-1}+2 = -2$
  $a^{-1} = -2-a-2$
  $a^{-1} = -a-4$
  $a^{-1}.a = -2 : a^{-1}.a = (-a-4).a = (-a-4)+a+2 = -4+2 = -2$
  Terpenuhi invers.

4 aksioma terpenuhi maka $(\mathbb{R},*)$ adalah grup

Translate
four group axioms
- **Closed**
  $a.b = a+b+2$
  a is a real number
  b is a real number
  2 is a real number
  So, then $(a+b+2)$ will result in a real number
  therefore $a.b \in \mathbb{R}$
- **Identity element**
  $a.e = a$
  $a+e+2 = a$
  $e+2 = 0$
  $e = -2$
  $e.a = -2.a = (-2)+a+2 = a$
  the identity element is satisfied

Associative
$a,b,c \in \mathbb{R}, (a.b).c = a.(b.c)$
$(a.b).c = (a+b+2).c$
$= (a+b+2)+c+2$
$= a+b+c+4$
$a.(b.c) = a.(b+c+2)$
$= a+(b+c+2)+2$
$= a+b+c+4$
left-hand side = right-hand side, so associative
Inverse
$e = -2$
$a^{-1} \Rightarrow a.a^{-1} = -2$
$a^{-1} = -2-a-2$
$a^{-1} = -a-4$
$a^{-1}.a = (-a-4).a$
$= (-a-4)+a+2$
$= -4+2 = -2$
the inverse is satisfied

All axioms are satisfied, therefore $(\mathbb{R},*)$ is a group

Figure 1. S6's response in proving the mathematical statement

Subsequently, an interview was conducted with S6, guided by the characteristics of the written response. The following is a summary of the interview between the Researcher (R) and participant S6:

R : You wrote that $(\mathbb{R},*)$ with the operation "*" defined on $\mathbb{R}$ by the rule $a * b = a + b + 2$ is a group. Could you explain how you proved this?

S6 : At first, I looked at the given operation, $a * b = a + b + 2$, and I recalled that to prove a set with a given operation is a group, I must check four properties: closure, associativity, existence of an identity, and existence of inverses.

R : Why did you choose to check those four properties?

S6 : Because that is the definition of a group. If all the properties are satisfied, then the set with that operation is indeed a group. So, I started by checking those properties one by one.

R : Could you describe how you found the identity element?

S6 : I tried to find an element eee such that $a * e = a$. By working this out, I obtained the identity element as -2, which is also a element of $\mathbb{R}$.

R : How about when determining the inverse?

S6 : I worked out the form $a * a^{-1} = -2$. Then I obtained $a^{-1} = -4 - a$, which is also a element of $\mathbb{R}$. Thus, every element of the set has an inverse.

R : So, what is your conclusion about the given set?

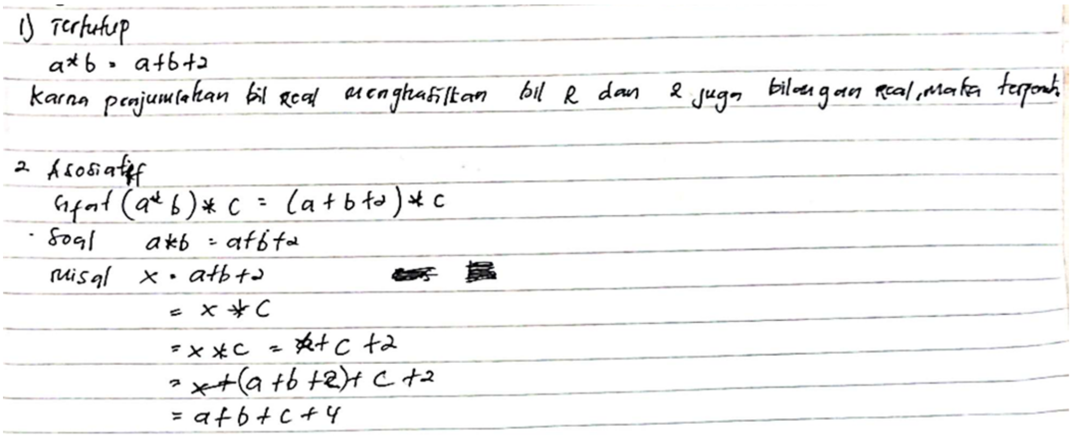
S6 : It is a group because it satisfies all the conditions.

In general, during the proving process S6 conjectured that $(\mathbb{R},*)$ might be a group. This conjecture was then confirmed through deductive reasoning based on the group definition. Prospective mathematics teachers in this group reached the correct conclusion that $(\mathbb{R},*)$ with the operation "*" defined on $\mathbb{R}$ by the rule $a * b = a + b + 2$ is indeed a group. For instance, S6 carried out the proof steps by showing that $a + b + 2$ also in $\mathbb{R}$, $(a * b) * c = a * (b * c)$ (associativity), that there exists $e = -2 \in \mathbb{R}$ such that $a * e = e * a = a$, and that there exists $a^{-1} = -4 - a \in \mathbb{R}$ such that $a * a^{-1} = a^{-1} * a = e$. The type of reasoning employed by prospective mathematics teachers in this group is categorized as Fact Optimization, since they formed conjectures based on available facts and then tested them deductively to reach a valid conclusion.

The study revealed that prospective mathematics teachers predominantly employed fact optimization in both proof construction and refutation tasks. This indicates a systematic tendency to formulate conjectures and verify them deductively, supported by procedural competence, which in turn facilitates valid proofs ((Firdausy et al., 2021; Jeannotte & Kieran, 2017; Nurafni et al., 2019). Abductive reasoning assists students in forming claims that can be transformed into deductive arguments during proof construction (Kaplan et al., 2021). These findings are consistent with Hidayah, et. al. (2020) who emphasized that fact optimization represents an ideal form of abductive reasoning, since the initial conjecture (hypothesis generation) is supported by correct facts and subsequently confirmed through proof. Such reliance on procedural approaches (Badjeber et al., 2025) highlights their difficulty in generating novel and relevant ideas, suggesting the need for instructional designs that foster more creative reasoning.

2. Abductive Reasoning Mistaken Fact by S4

S4 demonstrated knowledge of all the facts provided in the problem, indicating an understanding of both the given information and what needed to be proven. S4 applied the definition that a set is a group if it satisfies the four required properties. This is illustrated in Figure 2 below.



1) Tertutup

$a*b = a+b+2$

Karna penjumlahan bil real menghasilkan bil R dan 2 juga bilangan real, maka terpenuh

2 Asosiatif

Sifat $(a*b)*c = (a+b+2)*c$

· Soal $a*b = a+b+2$

Misal $x = a+b+2$

$= x*c$

$= x*c = \cancel{x}+c+2$

$= \cancel{x+}(a+b+2)+c+2$

$= a+b+c+4$

Translate

- **Closed**

  $a * b = a + b + 2$

  because the sum of real numbers is a real number and 2 is also a real number, it is satisfied
- **Associative**

  Propeties $(a * b) * c = (a + b + 2) * c$,

  Question $a * b = a + b + 2$

  Suppose $x = a + b + 2$

  $= x * c$

  $= x * c = x + c + 2$

  $= (a + b + 2) + c + 2$

  $= a + b + c + 4$

Figure 2. S4's response in proving the mathematical statement

The following excerpt presents the interview between the researcher (R) and participant S4

R : You solved the problem using the definition of a group. Could you explain the properties that must be satisfied $(\mathbb{R},*)$ to be a group?

S4 : Yes, to be considered a group, a set must satisfy four properties: closure, associativity, the existence of an identity, and the existence of inverses

R : Then how did you check the associative property for the operation $a * b = a + b + 2$?

S4 : I calculated $(a * b) * c$, and the result was $a + b + c + 4$

R : After obtaining $a * b = a + b + c + 4$, what did you do next?

S4 : I thought the result already clearly showed that the operation is associative, so I concluded that the associative property was satisfied.

R : You did not work out the form $a * (b * c)$

S4 : No, I didn't think it was necessary. I assumed the result would be the same, so I went straight to the conclusion.

R : So, what is your final conclusion about the given set?

S4 : It is a group.

In solving the problem, S4 demonstrated the associative property by expanding $(a * b) * c = a + b + c + 4$. However, S4 concluded that the operation on $(\mathbb{R},*)$ satisfies associativity without verifying that $(a * b) * c = a * (b * c)$. Thus, although the conclusion appeared correct, the reasoning process was flawed, making the final conclusion invalid. The type of reasoning used by students in this group falls under the category of mistaken fact, as they relied on an incorrect assumption as though it were a fact and used it as the basis for drawing conclusions.

The identification of mistaken fact reasoning highlights a critical weakness in prospective teachers' transition from procedural solutions to formal proof. Although they often recognized the direction of a solution, their reliance on unverified assumptions—such as treating partial expansions as evidence of associativity—reveals a fragile understanding of deductive verification. Such students tended to equate procedural solutions with formal proof, neglecting the deductive reasoning and logical justification required at each step (Badjeber et al., 2025). This tendency aligns with prior research, which indicates that prospective teachers often substitute informal arguments, examples, or generalizations for rigorous proof (Kandaga et al., 2022; Nadlifah & Prabawanto, 2017). The persistence of such reasoning errors underscores not only incomplete conceptual understanding (Stylianides, 2007), but also systemic challenges in fostering habits of universal justification. These findings suggest that strengthening deductive verification skills is essential and that instructional interventions should explicitly target the gap between procedural fluency and logically valid argumentation.

Prospective Mathematics Teachers' Abductive Reasoning Strategies in Refuting Mathematical Statements

1. Abductive Reasoning Fact Optimization by S1

S1 began solving the problem with the conjecture that not every subset of a group is necessarily a subgroup. S1 then applied the correct facts from the definition of a subgroup to test this conjecture and presented a counterexample as deductive proof to support the refutation. The counterexample used was a global, as it refuted the universal claim stated in the problem. S1's written response is presented in Figure 3 below.

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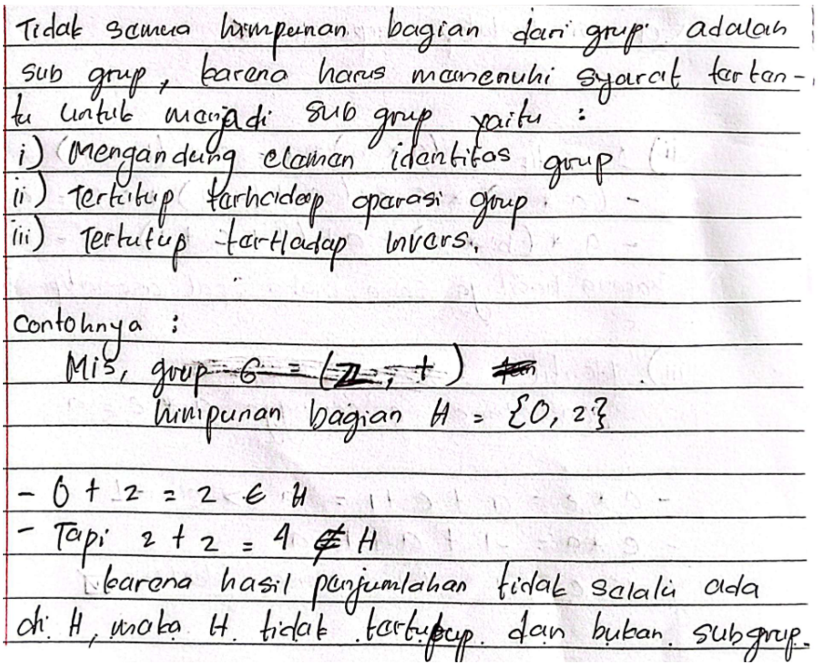

Tidak samua himpunan bagian dari grup adalah sub grup, karena harus mamenuhi syarat tertentu untuk menjadi sub grup yaitu :

i) Mengandung elaman identitas grup
ii) Tertutup terhadap oparasi grup
iii) Tertutup terhadap invers.

Contohnya :

Mis, ~~grup G = (Z, +)~~
himpunan bagian H = {0, 2}

- 0 + 2 = 2 ∈ H
- Tapi 2 + 2 = 4 ∉ H

karena hasil penjumlahan tidak selalu ada di H, maka H tidak tertutup. dan bukan subgrup.

Translate

Not every subset of a group is a subgroup, because it must satisfy certain conditions to be a subgroup, namely:

i) it contains the identity element of the group
ii) it is closed under the group operation
iii) it is closed under taking inverses

for example, consider the subset $H = (0,2)$

$0 + 2 = 2 \in H$,

but $2 + 2 = 4 \notin H$

since the result of the operation is not always in H, then H is not closed and therefore is not a subgroup

Figure 3. S1's response in refuting the mathematical statement

The following section presents an excerpt from the interview between the researcher (R) and subject S1

R	:	You wrote at the beginning of your answer that not every subset of a group is a subgroup. Could you explain the reason behind that?
S1	:	Yes, to be considered a group, a set must satisfy four properties: closure, associativity, the existence of an identity, and the existence of inverses
R	:	What are the conditions that must be satisfied for a subset to be considered a subgroup?
S1	:	At least three main conditions must be met: the set must be non-empty, closed under the group's binary operation, contain the identity element, and every element must have an inverse. If even one of these conditions is not fulfilled, then it cannot be considered a subgroup.

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R : You mentioned an example showing that the claim in the problem does not hold. Could you explain the example you used?

S1 : I chose $H = \{0,2\}$ y which is a subset of $\mathbb{Z}$. Since $\mathbb{Z}$ with the operation of addition forms a group, I used that as the context.

R : How did you check whether H is a subgroup or not?

S1 : I only checked the closure property. After checking, I found that H is not closed under addition. This means the subgroup condition is not satisfied. Since even one condition is not met, the set cannot be considered a subgroup.

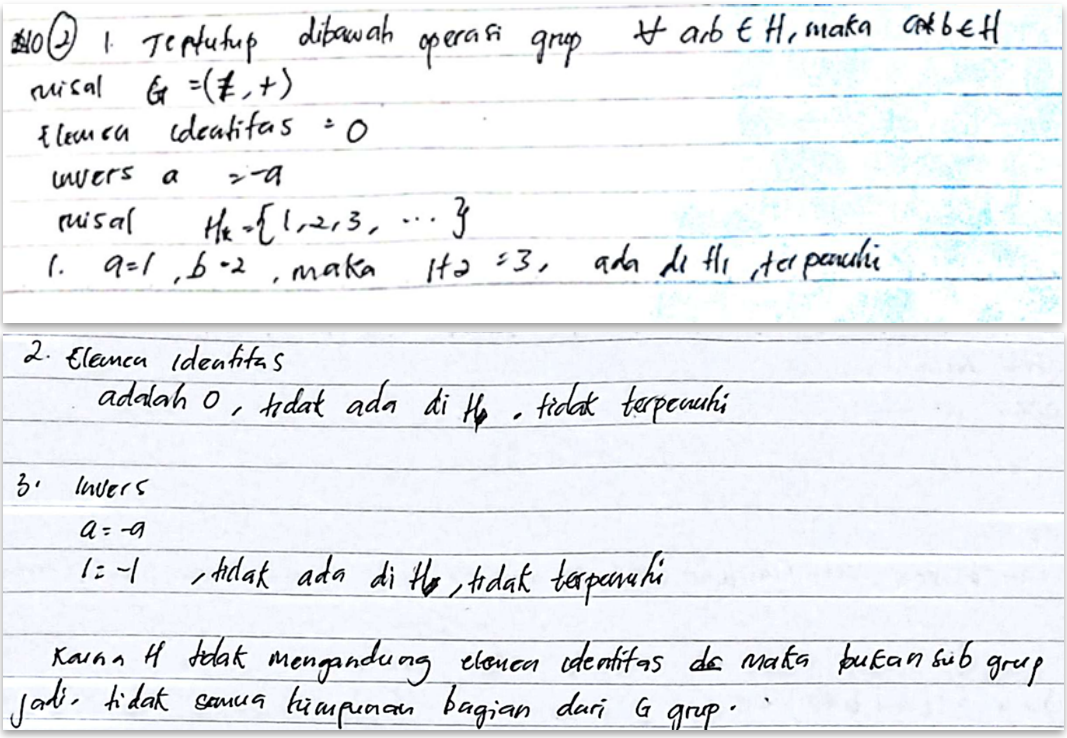
In testing the conjectures, S1 employed a counterexample using the set $H = \{0,2\}$ with the operation of addition on integers, which constitutes a subset of the group of integers under addition $(\mathbb{Z}, +)$.. Subsequently, the prospective mathematics teachers examined the subgroup conditions. The process began with checking the closure property through the following operation.

$$2 + 2 = 4 \notin H$$

This fact indicates that H s not closed under the operation of addition. Since it fails to satisfy even one of the required conditions, H cannot be considered a subgroup of $(\mathbb{Z}, +)$. S1 then related this finding to the general conclusion that not every subset of a group is a subgroup, as demonstrated by the provided counterexample. This response narrative shows that the prospective teachers optimized existing factual knowledge, namely the definition of a subgroup, and subsequently confirmed it through deductive reasoning using a counterexample. This process aligns with the characteristics of Fact Optimization, which involves formulating a conjectures based on factual knowledge, followed by systematic testing to arrive at a valid conclusion. However, the prospective mathematics teachers did not employ creative abduction to address the counterexample they identified in disproving the given mathematical statement (Komatsu & Jones, 2022) rather they engaged in a form of reasoning characterized as fact optimization.

2. Abductive Reasoning Mistaken Fact by S4

S4 approached the problem using the definition of a subgroup and then selected a global counterexample to check the fulfillment of each property specified by the definition. S4's written response is presented in Figure 4 below.



No ② 1. Tertutup dibawah operasi grup $\forall a,b \in H$, maka $a*b \in H$
misal $G = (\mathbb{Z}, +)$
Elemen identitas $= 0$
invers $a = -a$
misal $H_1 = \{1, 2, 3, \dots\}$
1. $a=1$, $b=2$, maka $1+2=3$, ada di H_1, terpenuhi

2. Elemen identitas
adalah 0, tidak ada di H_1. tidak terpenuhi

3. Invers
$a = -a$
$1 = -1$, tidak ada di H_1, tidak terpenuhi

Karena H tidak mengandung elemen identitas dia maka bukan sub grup
Jadi, tidak semua himpunan bagian dari G grup.

Translate

1. **Closed** under the group operation, $\forall a, b \in H$, so $a * b \in H$.
   Suppose $G = (\mathbb{Z}, +)$
   The identity element = 0.
   Inverse $a = -a$.
   Suppose $H_1 = \{1, 2, 3 \dots .\}$.
   $a = 1, b = 2$, so $1 + 2 = 3$, is in H_1 so the condition is satisfied
2. **The identity element**
   Is 0, is not in H_1 so the condition is not satisfied
3. **Inverse**
   $a = -a$
   $1 = -1$, is not in H_1 so the condition is not satisfied
   because H does not contain the identity element, it is not a subgroup
   therefore, not every subset of G is a group

Figure 4. S4's response in refuting the mathematical statement

The following excerpt presents the results of an interview between the researcher (R) and subject S4.

R : You stated that not every subset of a group is a subgroup. Could you explain how you verified that statement?

S4 : I used the definition of a subgroup. A subset must satisfy the properties of being non-empty, closed under the group operation, containing the identity element, and having inverses for all its elements in order to be considered a subgroup. To refute the general statement about subsets, I chose $H = \{1, 2, 3, \ldots..\}$, which is a subset of the group of integers under addition.

R : Why did you choose $H = \{1, 2, 3, \ldots..\}$ as your example?

S4 : Because it is clearly a subset of $\mathbb{Z}$, but I suspected that not all subgroup conditions would be satisfied in this case.

R : How did you check the closure property for H?

S4 : I tried adding $1 + 2 = 3$, and since the result is still in H. So, I concluded that H is closed under addition.

R : Do you think one example is sufficient to confirm that H is closed?

S4 : Yes, I thought it was sufficient, because if one example works, the others should as well.

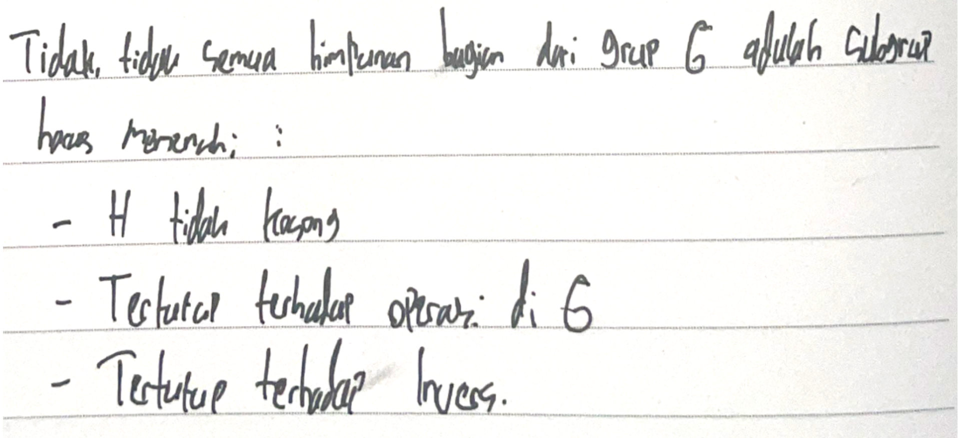
R : So, what was your final conclusion about gH?

S4 : It is not a subgroup because it does not contain the identity element and its members do not have inverses.

In solving the problem, S4 employed a counterexample using the set $H = \{1, 2, 3, \ldots..\}$ or the set of positive integers under addition, which is a subset of the group of integers with the operation of addition $(\mathbb{Z}, +)$. S4 demonstrated closure by illustrating the operation $1 + 2 = 3 \in H$. However, this process does not actually establish the closure property in a universal sense. Consequently, although S4's conclusion appeared to be correct on the surface, it was based on flawed reasoning, leading to an incorrect conclusion. The type of reasoning used by the prospective mathematics teacher in this group is identified as mistaken fact reasoning, characterized by the use of an incorrect assumption treated as if it were a fact and used as the basis for drawing a conclusion.

3. Abductive Reasoning Factual Error by S3

S3 began solving the problem with the claim that not every subset of a group is a subgroup. However, S3 relied on incomplete facts stating that a subset H of a group G is a subgroup if it is non-empty, closed under the binary operation in G, and has inverses. One essential property—the existence of an identity element in H - was not mentioned. S3's written response is presented in Figure 5 below.



Tidak, tidak semua himpunan bagian dari grup G adalah subgrup harus memenuhi; :

- H tidak kosong
- Tertutup terhadap operasi di G
- Tertutup terhadap Invers.

Translate

No, not every subset of a group Gis a subgroup; it must satisfy certain conditions.
H is non-empty
closed under the operation in G
closed under taking inverses

Figure 5. S3's response in refuting the mathematical statement

The following excerpt presents the results of an interview between the researcher (R) and subject S3

R	:	You wrote that not every subset of a group is a subgroup. Could you explain why?
S3	:	Because a subset can only be called a subgroup if it satisfies certain conditions. So, not every subset automatically qualifies as a subgroup.
R	:	What are the conditions that must be met for a subset to be considered a subgroup?
S3	:	As far as I know, the conditions are that the set must be non-empty, closed under the group's binary operation, and that every element has an inverse?
R	:	What about the condition that a subgroup must contain the identity element?
S3	:	I think it doesn't need to be mentioned, sir. As long as the set has inverses and is closed, I believe that's sufficient for it to be a subgroup.

S3 began solving the problem by claiming that not every subset of a group is a subgroup. However, when explaining the subgroup criteria, S3 only mentioned that the subset must be non-empty, closed under the group's binary operation, and that each element must have an inverse within the subset. S3 did not include the requirement of an identity element within the subset. This response reflects a misconception, as S3 assumed that the identity condition did not need to be checked or verified. This situation is categorized as factual error reasoning, since the prospective mathematics teachers relied on incorrect or incomplete facts to solve the problem.

The presence of prospective mathematics teachers exhibiting factual error reasoning in the refutation process also deserves attention. These prospective mathematics teachers relied on incomplete facts, as they omitted the requirement of the identity element in defining a subgroup. Consequently, their understanding of subgroup criteria was incomplete, reflecting a misconception of the formal definition. Errors in understanding mathematical concepts, definitions, or procedures are often rooted in shallow or limited comprehension (Elfiah et al., 2020). Abakah dan Brijlall (2024) further argue that misconceptions in mathematics arise from students' inability to recall and fully grasp concepts. Such misconceptions of previously learned material may lead to systematic errors in constructing mathematical arguments (Faisal et al., 2024). Therefore, explicit emphasis on the use of formal definitions in instruction is essential, so that prospective mathematics teachers not only recall them but are also able to integrate definitions fully into the processes of proof and refutation.

In the process of proving and refuting mathematical statements, abductive reasoning plays a crucial role, as it enables individuals to generate new ideas and innovative solutions by proposing plausible explanations (Widadah et al., 2024). An effective abductive strategy helps learners analyze and apply essential concepts relevant to given premises, thereby forming intermediate targets before concluding (Salsabila et al., 2021; Shodikin, 2017). Through abductive reasoning, students can identify and analyze emerging patterns in complex mathematical situations (Cramer-Petersen et al., 2019) which in turn supports them in constructing claims that can later be transformed into deductive arguments during proof processes (Kaplan et al., 2021). This deepens their understanding of the relationships among variables, algebraic structures, and mathematical properties (Widadah et al., 2024).

The absence of prospective mathematics teachers exhibiting creative conjecture characteristics also provides an interesting insight. According to Hidayah et al. (2020), this type represents the most creative form of abductive reasoning, as students not only utilize existing facts but also construct alternative possibilities that can form new contexts for proof. This finding suggests that students tend to rely on algorithmic and procedural thinking patterns (Badjeber et al., 2025), making them less inclined to develop alternative conjectures. Moreover, when task designs predominantly encourage the use of definitional rules rather than strategic rules, they can hinder both proving and refuting processes (Pedemonte, 2018). In such cases, students tend to focus on verifying existing conditions instead of formulating new hypotheses. Open-ended tasks, on the other hand, can foster the development of creative conjectures (Komatsu & Jones, 2022).

Overall, the findings of this study make a significant contribution to understanding the types of abductive reasoning demonstrated by prospective mathematics teachers in proving and refuting mathematical statements. The dominance of fact optimization indicates a positive potential for constructing formal proofs, as it is one of the most relevant forms of abductive reasoning for problem-solving (Hidayah et al., 2020). Thus, prospective mathematics teachers' abductive reasoning contributes to formulating mathematical conjectures and subsequently verifying them through deductive reasoning. However, the presence of mistaken facts and factual error reasoning emphasizes the need for instructional interventions that focus on completeness of definitions, clarity of deductive processes, and reflection on prospective mathematics teachers' reasoning errors. Furthermore, the absence of creative conjecture reasoning underscores the need for pedagogical strategies that encourage creative idea exploration. Irrelevant instructional practices may lead prospective mathematics teachers to develop weak conceptual connections, preventing them from meaningfully understanding definitions or applying them in new contexts (Chand, 2021). Therefore, the results of this study provide a foundation for designing more innovative learning approaches aimed at enhancing prospective mathematics teachers' proving and refuting abilities.

Integrating abductive reasoning into the curriculum can enhance prospective mathematics teachers' capabilities in constructing mathematical proofs (Widadah et al., 2024). This aligns with Hidayah et al.(2020) who stated that the type of abductive reasoning may vary depending on the problem context and instructional intervention. The role of educators—both teachers and lecturers—is crucial in helping learners select appropriate rules for problem solving, including in proving and refuting processes (Pedemonte & Reid, 2011). By understanding the types of abductive reasoning students employ, instructors can design targeted interventions to correct reasoning errors and provide more learning opportunities that promote continuous knowledge construction (Hidayah et al., 2020).

Although this study offers insights into the characteristics of prospective mathematics teachers' abductive reasoning in proving and refuting mathematical statements, several limitations should be acknowledged. First, the relatively small sample size and limited research context (within a single class and specific course) restrict the generalizability of the findings. Second, the number and variety of tasks used may not have fully captured the entire range of abductive reasoning types. Additionally, this study did not explore in depth the characteristics of counterexamples produced by students during refutation. The analysis focused solely on identifying the types of counterexamples used, without comprehensively examining the rationale behind their selection.

CONCLUSIONS AND RECOMMENDATIONS

This study revealed that prospective mathematics teachers employed several types of abductive reasoning strategies in proof and refutation. In proof construction, two strategies emerged: fact optimization, where they used both given and valid external facts (e.g., group definition) to form and confirm conjectures deductively, and mistaken fact, where incorrect assumptions led to seemingly correct but ultimately invalid conclusions. In refutation tasks, three strategies were identified: fact optimization, involving correct use of subgroup definitions supported by global counterexamples; mistaken fact, where reasoning omitted universal validation of subgroup properties; and factual error, where incomplete facts (omitting the identity element) resulted in inaccurate claims about subgroup conditions.

This study, limited to a single topic and a small set of problems, restricts the generalizability of its findings. Future research should employ broader contexts and more comprehensive instruments to capture diverse forms of abductive reasoning. Strengthening prospective mathematics teachers' abductive reasoning and exploring supportive learning environments are essential directions for further study.

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