ANALYSIS OF THE LINEAR NAVIER STOKES KORTEWEG MODEL WITH NEUMANN BOUNDARY CONDITIONS IN THREE DIMENSIONAL HALF-SPACE

Dwi Windari Nur Khasanah[1], Suma Inna[2], Madona Yunita Wijaya[3], Sri Indriati Hasanah[4]
[1,2,3] Universitas Islam Negeri Syarif Hidayatullah Jakarta, Indonesia
[4] Universitas Madura, Pamekasan, Jawa Timur, Indonesia
dwiwindari710@gmail.com

ABSTRACT This study discusses the solution of the Navier-Stokes Korteweg model, which describes two-phase fluid flow with capillary effects, with Neumann boundary conditions in the half-space. The main objective is to detail the resolution process of the resolvent equation system in the half-space related to the Navier-Stokes Korteweg model with Neumann boundary conditions. The resolution is carried out in several steps. First, the resolvent equation system is reduced using even and odd extensions. Then, a partial Fourier transform is applied, resulting in a simpler ordinary differential equation. The findings of this research indicate the existence of a solution operator for the resolvent equation of the Navier-Stokes Korteweg model with Neumann boundary conditions in the half-space. This solution applies for two cases involving the coefficients, depending on certain conditions related to the fluid properties.

Keywords: extensions, navier stokes korteweg, neumann, resolvent equation, partial fourier transformation.

ABSTRAK Penelitian ini membahas solusi dari model Navier-Stokes Korteweg, yang menggambarkan aliran fluida dua fase dengan efek kapiler, dengan kondisi batas Neumann di ruang setengah. Tujuan utama dari penelitian ini adalah merinci proses penyelesaian sistem persamaan resolven di ruang setengah yang terkait dengan model Navier-Stokes Korteweg dengan kondisi batas Neumann. Penyelesaiannya dilakukan melalui beberapa langkah. Pertama, sistem persamaan resolven direduksi menggunakan ekstensi genap dan ekstensi ganjil. Kemudian, transformasi Fourier parsial diterapkan, menghasilkan persamaan diferensial biasa yang lebih sederhana. Hasil penelitian ini menunjukkan adanya operator solusi untuk persamaan resolven model Navier-Stokes Korteweg dengan kondisi batas Neumann di ruang setengah. Solusi ini berlaku untuk dua kasus yang melibatkan koefisien, tergantung pada kondisi tertentu yang berkaitan dengan sifat fluida.

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Keywords: eksistensi, navier stokes korteweg, neumann, persamaan resolvent, transformasi fourier parsial.

INTRODUCTION

All substances that can flow, whether as a gas or a liquid, fall into the category pf fluids. The flow properties of fluids, both liquids and gases, constantly change when subjected to shear stress, giving them the characteristic flowability known as "fluidity" Ridwan (1999). Fluids can be classified into two types based on their response to pressure: compressible and incompressible. Gases, as compressible fluids, experience significant changes in density when subjected to pressure, while liquids, which are incompressible, have minimal changes in density, making it effectively constant Pritchard (2011).

One of the mathematical models used to describe compressible fluid flow is the Navier Stokes Korteweg model.

$$\begin{aligned} \partial_t \rho + div(\rho \boldsymbol{u}) &= 0 && \text{in } \Omega \times (0, T) \\ \rho(\partial_t u + \boldsymbol{u} \cdot \nabla \boldsymbol{u}) &= Div(S(u) + K(\rho) - P(\rho)\mathbf{I}) && \text{in } \Omega \times (0, T) \end{aligned} \tag{1}$$

The Navier Stokes Korteweg equations is used to model the flow of fluids and gases undergoing phase transitions. This equation is an extension of the Navier Stokes equations, which serves as the foundation for describing compressible fluids flow, such as gases. The primary difference between the Navier Stokes Korteweg and Navier Stokes equations lies in the inclusion of the stress tensor and the capillarity constant. If the capillarity constant is set to 0, the Navier Stokes Korteweg equations becomes equivalent to the Navier Stokes equations.

Numerous studies have been conducted by previous researchers o fluid model solutions. Dunn & Serrin (1985) introduced the concept of interstitial work in thermodynamics and designed a Korteweg0type fluid model that accounts for the effects of the stress tensor. Subsequently, Desjardins & Danchin (2001) demonstrated the existence of a unique smooth solution for the compressible capillary isothermal fluid model, which can represent phase transitions. Hattori & Li (1996) also proved the existence of global solutions for the Korteweg syste in high dimensions when the initial data is small, which represents a simplified isothermal version. Haspot (2011) confirmed the existence of global weak solutions for general capillary isothermal fluid models, which serve as phase transition models. Additionally, Haspot (2011) improved upon Desjardins & Danchin (2001) results by showing the existence of global weak solutions in one dimension for certain types of capillarity coefficients with large initial data in energy spaces. Kotschote (2008) demonstrated the existence of a unique local solution for the isothermal fluid model.

Saito (2019) discussed a compressible Korteweg-type fluid model involving free boundary conditions and successfully developed an operator for a unique solution. Saito (2020) studied the existence of a family of R-bounded solution operators for the compressible Korteweg-type fluid model in $\mathbf{R}^3_+$ under the boundary conditions $\mathbf{n} \cdot \nabla \rho = 0$ on Γ and $\mathbf{u} = 0$ on Γ.

For each value of the coefficients $\mu_*, \nu_*,$ and κ_*, there are five cases that need to be analyzed, namely:

Case I: $\left(\frac{\mu_*+\nu_*}{2\kappa_*}\right)^2-\left(\frac{1}{\kappa_*}\right)<0$

Case II: $\left(\frac{\mu_*+\nu_*}{2\kappa_*}\right)^2-\left(\frac{1}{\kappa_*}\right)>0, \kappa_* \neq \mu_*\nu_*;$

Case III: $\left(\frac{\mu_*+\nu_*}{2\kappa_*}\right)^2-\left(\frac{1}{\kappa_*}\right)>0, \kappa_* = \mu_*\nu_*;$

Case IV: $\left(\frac{\mu_*+\nu_*}{2\kappa_*}\right)^2-\left(\frac{1}{\kappa_*}\right)=0, \kappa_* \neq \mu_*\nu_*;$

Case V: $\left(\frac{\mu_*+\nu_*}{2\kappa_*}\right)^2-\left(\frac{1}{\kappa_*}\right)=0, \kappa_* = \mu_*\nu_*.$

Inna, Maryani, & Saito (2020) reviewed the Korteweg-type fluid model in $\mathbf{R}^3_+$ with slip boundary conditions, namely $\mathbf{n}\cdot\nabla\rho = g, \partial_N u_j + \partial_j u_N = h_j$ and $u_N = h_N$. Subsequently, Inna, Fauziah, Manaqib, & Maya Putri (2023) studied the compressible Korteweg-type fluid model with slip boundary conditions in $\mathbf{R}^3_+$ for the case where the coefficient $\left(\frac{\mu+\nu}{2\kappa}\right)^2-\left(\frac{1}{\kappa}\right)\neq 0, \kappa = \mu\nu$, with $\mu \neq \nu$. Then, Inna & Saito (2023) analyzed the Navier Stokes Korteweg model involving the time variable t and demonstrated the existence of a local solution with slip boundary conditions. After that, Inna S. (2024) examined the existence of R-bounded solution operators in the Navier Stokes Korteweg model with slip boundary conditions in $\mathbf{R}^3_+$. Additionally, Prayugo, Inna, Mahmudi, & Damiati (2024) discussed the solution of the linear Navier Stokes Korteweg model in $\mathbf{R}^3_+$ with slip boundary conditions for the case where the coefficient $\left(\frac{\mu+\nu}{2\kappa}\right)^2-\left(\frac{1}{\kappa}\right)\neq 0, \kappa=\mu\nu, \mu\neq\nu$. Similarly, Salsabila, Inna, Liebenlito, & Purnomowati (2024) discussed the solution of the Navier Stokes Korteweg model with slip boundary conditions in a three-dimensional for two coefficient cases: $\left(\frac{\mu+\nu}{2\kappa}\right)^2-\left(\frac{1}{\kappa}\right)<0$ and $\left(\frac{\mu+\nu}{2\kappa}\right)^2-\left(\frac{1}{\kappa}\right)>0, \kappa\neq\mu\nu$.

Meanwhile, in this study, the author examines in detail the determination of the solution operator for the resolvent equation of the Navier Stokes Korteweg model in the half-space ($\mathbf{R}^3_+$) for two coefficient cases, namely $\left(\frac{\mu_*+\nu_*}{2\kappa_*}\right)^2-\left(\frac{1}{\kappa_*}\right)<0$ and $\left(\frac{\mu_*+\nu_*}{2\kappa_*}\right)^2-\left(\frac{1}{\kappa_*}\right)>0, \kappa_*\neq\mu_*\nu_*$ respectively.

METHODS

In general, solving the nonlinear equations (1) with boundary conditions is done through several steps. First, the system is linearized. Then, a Laplace transform is applied to eliminate the time variable t. This process yields a system of equations known as the resolvent equation. The resolvent equation corresponding to equation (1) is as follow:

$$\lambda\rho + \frac{\partial}{\partial w_1}u_1 + \frac{\partial}{\partial w_2}u_2 + \frac{\partial}{\partial w_3}u_3 = d \qquad \text{in } \mathbf{R}^3$$

$$\begin{aligned}\lambda u_1 - \mu_*\left(\frac{\partial^2}{\partial {w_1}^2}+\frac{\partial^2}{\partial {w_2}^2}+\frac{\partial^2}{\partial {w_3}^2}\right)u_1 - \nu_*\frac{\partial}{\partial w_1}\left(\frac{\partial}{\partial w_1}u_1+\frac{\partial}{\partial w_2}u_2+\frac{\partial}{\partial w_3}u_3\right) - \\ \kappa_*\frac{\partial}{\partial w_1}\left(\frac{\partial^2}{\partial {w_1}^2}+\frac{\partial^2}{\partial {w_2}^2}+\frac{\partial^2}{\partial {w_3}^2}\right)\rho = f_1\end{aligned} \qquad \text{in } \mathbf{R}^3 \qquad (2)$$

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$$\lambda u_2 - \mu_* \left(\frac{\partial^2}{\partial {w_1}^2} + \frac{\partial^2}{\partial {w_2}^2} + \frac{\partial^2}{\partial {w_3}^2}\right) u_2 - \nu_* \frac{\partial}{\partial w_2}\left(\frac{\partial}{\partial w_1}u_1 + \frac{\partial}{\partial w_2}u_2 + \frac{\partial}{\partial w_3}u_3\right) - \kappa_* \frac{\partial}{\partial w_2}\left(\frac{\partial^2}{\partial {w_1}^2} + \frac{\partial^2}{\partial {w_2}^2} + \frac{\partial^2}{\partial {w_3}^2}\right)\rho = f_2 \quad \text{in } \mathbf{R}^3$$

$$\lambda u_3 - \mu_* \left(\frac{\partial^2}{\partial {w_1}^2} + \frac{\partial^2}{\partial {w_2}^2} + \frac{\partial^2}{\partial {w_3}^2}\right) u_3 - \nu_* \frac{\partial}{\partial w_3}\left(\frac{\partial}{\partial w_1}u_1 + \frac{\partial}{\partial w_2}u_2 + \frac{\partial}{\partial w_3}u_3\right) - \kappa_* \frac{\partial}{\partial w_3}\left(\frac{\partial^2}{\partial {w_1}^2} + \frac{\partial^2}{\partial {w_2}^2} + \frac{\partial^2}{\partial {w_3}^2}\right)\rho = f_3 \quad \text{in } \mathbf{R}^3$$

In this study, we will discuss the solution of equation (2) in the half-space ($\mathbf{R}^3_+$) with Neumaan boundary conditions, which is expressed as follows:

$$\lambda\rho + \frac{\partial}{\partial w_1}u_1 + \frac{\partial}{\partial w_2}u_2 + \frac{\partial}{\partial w_3}u_3 = d \quad \text{in } \mathbf{R}^3_+$$

$$\lambda u_1 - \mu_* \left(\frac{\partial^2}{\partial {w_1}^2} + \frac{\partial^2}{\partial {w_2}^2} + \frac{\partial^2}{\partial {w_3}^2}\right) u_1 - \nu_* \frac{\partial}{\partial w_1}\left(\frac{\partial}{\partial w_1}u_1 + \frac{\partial}{\partial w_2}u_2 + \frac{\partial}{\partial w_3}u_3\right) - \kappa_* \frac{\partial}{\partial w_1}\left(\frac{\partial^2}{\partial {w_1}^2} + \frac{\partial^2}{\partial {w_2}^2} + \frac{\partial^2}{\partial {w_3}^2}\right)\rho = f_1 \quad \text{in } \mathbf{R}^3_+$$

$$\lambda u_2 - \mu_* \left(\frac{\partial^2}{\partial {w_1}^2} + \frac{\partial^2}{\partial {w_2}^2} + \frac{\partial^2}{\partial {w_3}^2}\right) u_2 - \nu_* \frac{\partial}{\partial w_2}\left(\frac{\partial}{\partial w_1}u_1 + \frac{\partial}{\partial w_2}u_2 + \frac{\partial}{\partial w_3}u_3\right) - \kappa_* \frac{\partial}{\partial w_2}\left(\frac{\partial^2}{\partial {w_1}^2} + \frac{\partial^2}{\partial {w_2}^2} + \frac{\partial^2}{\partial {w_3}^2}\right)\rho = f_2 \quad \text{in } \mathbf{R}^3_+ \qquad (3)$$

$$\lambda u_3 - \mu_* \left(\frac{\partial^2}{\partial {w_1}^2} + \frac{\partial^2}{\partial {w_2}^2} + \frac{\partial^2}{\partial {w_3}^2}\right) u_3 - \nu_* \frac{\partial}{\partial w_3}\left(\frac{\partial}{\partial w_1}u_1 + \frac{\partial}{\partial w_2}u_2 + \frac{\partial}{\partial w_3}u_3\right) - \kappa_* \frac{\partial}{\partial w_3}\left(\frac{\partial^2}{\partial {w_1}^2} + \frac{\partial^2}{\partial {w_2}^2} + \frac{\partial^2}{\partial {w_3}^2}\right)\rho = f_3 \quad \text{in } \mathbf{R}^3_+$$

$$\begin{pmatrix}0\\0\\-1\end{pmatrix} \cdot \begin{pmatrix}\frac{\partial}{\partial w_1}\\ \frac{\partial}{\partial w_2}\\ \frac{\partial}{\partial w_3}\end{pmatrix}\rho = g, \quad \begin{pmatrix}u_1\\u_2\\u_3\end{pmatrix} = \begin{pmatrix}0\\0\\0\end{pmatrix} \quad \text{in } \mathbf{R}^3_0$$

Here, $\mathbf{n} = (0,0,-1)^\intercal$ is defined as the unit normal vector pointing outward from $\mathbf{R}^3_0$. The parameter λ is a complex number in $\mathbf{C}_+ = \{z \in \mathbf{C} | \Re_z > 0\}$. The right-hand side functions are given by $d = d(w_1, w_2, w_3), g = g(w_1, w_2, w_3)$, and $\mathbf{f} = \mathbf{f}(w_1, w_2, w_3) = \left(f_1(w_1, w_2, w_3), f_2(w_1, w_2, w_3), f_3(w_1, w_2, w_3)\right)^\intercal$, which are known functions. The vector-valued functions $\mathbf{u} = \mathbf{u}(w_1, w_2, w_3) = \left(u_1(w_1, w_2, w_3), u_2(w_1, w_2, w_3), u_3(w_1, w_2, w_3)\right)^\intercal$ represents the solution vector, while $\rho = \rho(w_1, w_2, w_3)$ is a scalar function representing the velocity and density of the fluid. The regions are defined as follows:

$$\mathbf{R}^3_+ = \{w = (w_1, w_2, w_3) \in \mathbf{R}^3, w_3 > 0\},$$
$$\mathbf{R}^3_0 = \{w = (w_1, w_2, w_3) \in \mathbf{R}^3, w_3 = 0\},$$

To solve equation (3) with boundary conditions in general, the steps include solving equation (3) in the whole space, then in half space, and finally in the bent half space.

Some special notations used in this article are as follows: $\mathbf{N}$ denotes the set of natural numbers, $\mathbf{C}$ denotes the set of complex numbers, and $\mathbf{R}$ denotes the set of real numbers. Let $q \in [1, \infty)$, then $L_q(\mathbf{R}^3_+)$ represents the Lebesgue space and $W^m_q(\mathbf{R}^3_+)$ is the Sobolev space in $\mathbf{R}^3_+$, with $m \in \mathbf{N}$. If $m = 0$, then $W^0_q(\mathbf{R}^3_+) = L_q(\mathbf{R}^3_+)$ and the norm in $W^n_q(\mathbf{R}^3_+), n \in \mathbf{N_0}, \mathbf{N_0} = \mathbf{N} \cup \{0\}$ is denoted by $\|\cdot\|_{W^n_q(\mathbf{R}^3_+)}$. Let X and Y be Banach spaces. Then X^m, for $m \in \mathbf{N}$ denotes the Cartesian product of X taken m times, and the norm in X^m is written as $\|\cdot\|_X$.

The set of linear operators from X to Y is denoted by $\mathcal{L}(X,Y)$ while the set of linear operators from X to X is denoted by $\mathcal{L}(X)$. For a domain $U \subset \mathbf{C}, Hol(U, \mathcal{L}(X,Y))$ denotes the set of holomorphic functions valued in $\mathcal{L}(X,Y)$ defined on U.

To solve equation (3), the solution approach for the system in the whole space $(\mathbf{R}^3)$ is given by the following equation,

$$
\begin{aligned}
&\lambda\rho + \frac{\partial}{\partial w_1}u_1 + \frac{\partial}{\partial w_2}u_2 + \frac{\partial}{\partial w_3}u_3 = d && \text{in } \mathbf{R}^3 \\
&\lambda u_1 - \mu_* \left(\frac{\partial^2}{\partial {w_1}^2} + \frac{\partial^2}{\partial {w_2}^2} + \frac{\partial^2}{\partial {w_3}^2}\right) u_1 - \nu_* \frac{\partial}{\partial w_1}\left(\frac{\partial}{\partial w_1}u_1 + \frac{\partial}{\partial w_2}u_2 + \frac{\partial}{\partial w_3}u_3\right) - \\
&\qquad \kappa_* \frac{\partial}{\partial w_1}\left(\frac{\partial^2}{\partial {w_1}^2} + \frac{\partial^2}{\partial {w_2}^2} + \frac{\partial^2}{\partial {w_3}^2}\right)\rho = f_1 && \text{in } \mathbf{R}^3 \\
&\lambda u_2 - \mu_* \left(\frac{\partial^2}{\partial {w_1}^2} + \frac{\partial^2}{\partial {w_2}^2} + \frac{\partial^2}{\partial {w_3}^2}\right) u_2 - \nu_* \frac{\partial}{\partial w_2}\left(\frac{\partial}{\partial w_1}u_1 + \frac{\partial}{\partial w_2}u_2 + \frac{\partial}{\partial w_3}u_3\right) - \\
&\qquad \kappa_* \frac{\partial}{\partial w_2}\left(\frac{\partial^2}{\partial {w_1}^2} + \frac{\partial^2}{\partial {w_2}^2} + \frac{\partial^2}{\partial {w_3}^2}\right)\rho = f_2 && \text{in } \mathbf{R}^3 \\
&\lambda u_3 - \mu_* \left(\frac{\partial^2}{\partial {w_1}^2} + \frac{\partial^2}{\partial {w_2}^2} + \frac{\partial^2}{\partial {w_3}^2}\right) u_3 - \nu_* \frac{\partial}{\partial w_3}\left(\frac{\partial}{\partial w_1}u_1 + \frac{\partial}{\partial w_2}u_2 + \frac{\partial}{\partial w_3}u_3\right) - \\
&\qquad \kappa_* \frac{\partial}{\partial w_3}\left(\frac{\partial^2}{\partial {w_1}^2} + \frac{\partial^2}{\partial {w_2}^2} + \frac{\partial^2}{\partial {w_3}^2}\right)\rho = f_3 && \text{in } \mathbf{R}^3
\end{aligned}
\tag{4}
$$

For equation (4), Saito (2019) obtained the following result,
Define the space $\mathbf{K}^0 = (d, f_1, f_2, f_3)$ as the right-hand side functions in equation (2) as follows.

$$\mathcal{X}_p^1(\mathbf{R}^3) = W_p^1(\mathbf{R}^3) \times L_p(\mathbf{R}^3) \times L_p(\mathbf{R}^3) \times L_p(\mathbf{R}^3)$$

Then, define $\mathfrak{X}_p^1(\mathbf{R}^3)$ and $\mathcal{K}_\lambda^0\mathbf{K}^0$ as follows:

$$\mathfrak{X}_p^1(\mathbf{R}^3) = L_p(\mathbf{R}^3)^A, \text{ where } A = (3+1)+3 = 7.$$

$$\mathcal{K}_\lambda^0\mathbf{K}^0 = \left(\frac{\partial}{\partial w_1}d, \frac{\partial}{\partial w_2}d, \frac{\partial}{\partial w_3}d, \lambda^{\frac{1}{2}}d, f_1, f_2, f_3\right) \in \mathfrak{X}_p^1(\mathbf{R}^3).$$

Next, the following theorem is obtained,
Theorem 1. Let $p \in (1,\infty)$ and assume that μ_*, ν_*, κ_* are positive constants satisfying $\left(\frac{\mu_*+\nu_*}{2\kappa_*}\right)^2 - \left(\frac{1}{\kappa_*}\right) < 0$ and $\left(\frac{\mu_*+\nu_*}{2\kappa_*}\right)^2 - \left(\frac{1}{\kappa_*}\right) > 0, \kappa_* \neq \mu_*\nu_*$. Then, for every $\lambda \in \boldsymbol{C}_+$, there exist operators $\mathfrak{A}^1(\lambda)$ and $\mathfrak{B}^1(\lambda)$ with,

$$\mathfrak{A}^1(\lambda) \in Hol\left(\boldsymbol{C}_+, \mathcal{L}\left(\mathfrak{X}_p^1(\mathbf{R}^3), W_q^3(\boldsymbol{R}^3)\right)\right),$$

$$\mathfrak{B}^1(\lambda) \in Hol\left(\boldsymbol{C}_+, \mathcal{L}\left(\mathfrak{X}_p^1(\mathbf{R}^3), W_q^2(\boldsymbol{R}^3)^3\right)\right),$$

such that, for any $\boldsymbol{K}^0 = (d, \mathbf{f}) \in \mathfrak{X}_p^1(\mathbf{R}^3)$, there exists a unique solution to the system (4) is $(\rho, \boldsymbol{u}) = \left(\mathfrak{A}^1(\lambda)\mathcal{K}_\lambda^0\mathbf{K}^0, \mathfrak{B}^1(\lambda)\,\mathcal{K}_\lambda^0\mathbf{K}^0\right)$.

FINDING AND DISCUSSION

This section serves as the main objective of this research, which is to demonstrate the existence of a solution for equation (3). The space $\mathbf{K}^1 = (d, f_1, f_2, f_3, g)$ is define as the right-hand side functions of equation (3), as follows:

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$$\mathcal{X}_p^2(\mathbf{R}_+^3) = W_p^1(\mathbf{R}_+^3) \times L_p(\mathbf{R}_+^3) \times L_p(\mathbf{R}_+^3) \times L_p(\mathbf{R}_+^3) \times W_p^2(\mathbf{R}_+^3)$$

Then, $\mathcal{K}_\lambda^0\mathbf{K}^1$ and $\mathfrak{X}_p^2(\mathbf{R}_+^3)$ are expressed as follows:

$$\mathfrak{X}_q^2(\mathbf{R}_+^3) = L_p(\mathbf{R}_+^3)^B,$$

$$B = (N+1) + N + (N^2 + N + 1) = 20.$$

$$\mathcal{K}_\lambda^0\mathbf{K}^1 = \left(\left(\frac{\partial}{\partial w_1}d, \frac{\partial}{\partial w_2}d, \frac{\partial}{\partial w_3}d, \lambda^{\frac{1}{2}}d\right), f_1, f_2, f_3,\right.$$

$$\left.\left(\begin{pmatrix} \frac{\partial^2}{\partial {w_1}^2}(g) & \frac{\partial^2}{\partial w_1 \partial w_2}(g) & \frac{\partial^2}{\partial w_1 \partial w_3}(g) \\ \frac{\partial^2}{\partial w_1 \partial w_2}(g) & \frac{\partial^2}{\partial {w_2}^2}(g) & \frac{\partial^2}{\partial w_2 \partial w_3}(g) \\ \frac{\partial^2}{\partial w_1 \partial w_3}(g) & \frac{\partial^2}{\partial w_2 \partial w_3}(g) & \frac{\partial^2}{\partial {w_3}^2}(g) \end{pmatrix}, \lambda^{\frac{1}{2}}\frac{\partial}{\partial w_1}g, \lambda^{\frac{1}{2}}\frac{\partial}{\partial w_2}g, \lambda^{\frac{1}{2}}\frac{\partial}{\partial w_3}g, \lambda g\right)\right) \in \mathfrak{X}_q^2(\mathbf{R}_+^3).$$

The main objective of this research is to obtain the solution operator for equation (3) by proving the Theorem 2.

Theorem 2. Let $q \in (1,\infty)$ and assume that μ, v, κ are positive constants satisfying $\left(\frac{\mu_* + v_*}{2\kappa_*}\right)^2 - \left(\frac{1}{\kappa_*}\right) < 0$ and $\left(\frac{\mu_* + v_*}{2\kappa_*}\right)^2 - \left(\frac{1}{\kappa_*}\right) > 0, \kappa_* \neq \mu_* v_*$. Then, for every $\lambda \in \boldsymbol{C}_+$, there exists $\mathfrak{A}^0(\lambda)$ and $\mathfrak{B}^0(\lambda)$ with,

$$\mathfrak{A}^0(\lambda) \in Hol\left(\boldsymbol{C}_+, \mathcal{L}\left(\mathfrak{X}_p^2(\boldsymbol{R}_+^3), W_q^3(\boldsymbol{R}_+^3)\right)\right),$$

$$\mathfrak{B}^0(\lambda) \in Hol\left(\boldsymbol{C}_+, \mathcal{L}\left(\mathfrak{X}_p^2(\boldsymbol{R}_+^3), W_q^2(\boldsymbol{R}_+^3)^3\right)\right),$$

such that, for any $\mathbf{K}^1 = (d, f_1, f_2, f_3, g) \in \mathcal{X}_p^2(\mathbf{R}_+^3)$, there exists a unique solution to the system (1) is $(\rho, \boldsymbol{u}) = \left(\mathfrak{A}^0(\lambda)\mathcal{K}_\lambda^0\mathbf{K}^1, \mathfrak{B}^0(\lambda)\,\mathcal{K}_\lambda^0\mathbf{K}^1\right)$.

Several steps are required to prove Theorem 2. First, the inhomogeneous system of equation (3) is reduced to a homogeneous system of equations. Then, the homogeneous system is solved.

Reduced System

The first step is to reduce the system of equation (1) using the solution operator approch in the whole space ($\mathbf{R}^3$) with even and odd extensions. For a function $f = f(w)$ with $w(w_1, w_2, w_3) \in \mathbf{R}_+^3$, the even extension $E^e f(w_1, w_2, w_3)$ and the odd extension $E^0 f(w_1, w_2, w_3)$ of f are defined as follows:

$$E^e f = (E^e f)(w_1, w_2, w_3) = \begin{cases} f(w_1, w_2, w_3), & (w_3 > 0) \\ f(w_1, w_2, -w_3), & (w_3 < 0) \end{cases}$$
$$E^0 f = (E^0 f)(w_1, w_2, w_3) = \begin{cases} f(w_1, w_2, w_3), & (w_3 > 0) \\ -f(w_1, w_2, -w_3), & (w_3 < 0) \end{cases} \quad (5)$$

Next, definie the extension of a vector function $\mathbf{f} = (f_1, f_2, f_3)^{\mathrm{T}}$ in $\mathbf{R}^3$ as follows:

$$\mathbf{Ef} = (E^e f_1, E^e f_2, E^0 f_3)^\intercal. \tag{6}$$

Note that $E^e \in \mathcal{L}\left(W_p^1(\mathbf{R}_+^3), W_p^1(\mathbf{R}^3)\right)$ and $\mathbf{E} \in \mathcal{L}\left(L_p(\mathbf{R}_+^3)^3, L_p(\mathbf{R}^3)^3\right)$.

Let $(\tilde{d}, \tilde{f})$ be a function in the space $W_p^1(\mathbf{R}_+^3) \times L_p(\mathbf{R}_+^3)^3$ in equation (3) and $\mathfrak{A}^1(\lambda)\mathcal{K}_\lambda^0\mathbf{K}^0$ and $\mathfrak{B}^1(\lambda)\mathcal{K}_\lambda^0\mathbf{K}^0$ are the solution operators presented in Theorem 1 in $\mathbf{R}^3$. Define the operators $\mathbf{M}$ and $\mathbf{P}$ as follows:

$$\mathbf{M} = \mathfrak{A}^1(\lambda)\mathcal{K}_\lambda^0\left(E^e\tilde{d}, \mathbf{Ef}\right) \text{ and } \mathbf{P} = \mathfrak{B}^1(\lambda)\mathcal{K}_\lambda^0\left(E^e\tilde{d}, \mathbf{Ef}\right). \tag{7}$$

Next, define $\mathbf{N} = \mathbf{N}(w_1, w_2, w_3)$ and $\mathbf{O} = \mathbf{O}(w_1, w_2, w_3)$ as follows:

$$\begin{aligned} \mathbf{N} = \mathbf{M}(w_1, w_2, -w_3), \text{and } \mathbf{O} \\ = \left(P_1(w_1, w_2, -w_3), P_2(w_1, w_2, -w_3), -P_3(w_1, w_2, -w_3)\right)^\intercal \end{aligned} \tag{8}$$

then,

$$O_3(w_1, w_2, w_3) = -P_3(w_1, w_2, w_3). \tag{9}$$

Substitute equation (8) into the first row of equation (3) to obtain:

$$\begin{aligned} &\left(\lambda N + \frac{\partial}{\partial w_1}O_1 + \frac{\partial}{\partial w_2}O_2 + \frac{\partial}{\partial w_3}O_3\right)(w_1, w_2, w_3) \\ &= \left(\lambda M + \frac{\partial}{\partial w_1}P_1 + \frac{\partial}{\partial w_2}P_2 + \frac{\partial}{\partial w_3}P_3\right)(w_1, w_2, -w_3) \\ &= (E^e d)(w_1, w_2, -w_3) \\ &= (E^e d)(w_1, w_2, w_3). \end{aligned} \tag{10}$$

To obtain the second, third, and fourth rows of equation (3):

$$\begin{aligned} &\left(\lambda O_1 - \mu_*\left(\frac{\partial^2}{\partial {w_1}^2} + \frac{\partial^2}{\partial {w_2}^2} + \frac{\partial^2}{\partial {w_3}^2}\right)O_1 - v_*\frac{\partial}{\partial w_1}\left(\frac{\partial}{\partial w_1}O_1 + \frac{\partial}{\partial w_2}O_2 + \frac{\partial}{\partial w_3}O_3\right) - \right. \\ &\left.\kappa_*\frac{\partial}{\partial w_1}\left(\frac{\partial^2}{\partial {w_1}^2} + \frac{\partial^2}{\partial {w_2}^2} + \frac{\partial^2}{\partial {w_3}^2}\right)N\right)(w_1, w_2, w_3) \\ &= \left(\lambda P_1 - \mu_*\left(\frac{\partial^2}{\partial {w_1}^2} + \frac{\partial^2}{\partial {w_2}^2} + \frac{\partial^2}{\partial {w_3}^2}\right)P_1 - v_*\frac{\partial}{\partial w_1}\left(\frac{\partial}{\partial w_1}P_1 + \frac{\partial}{\partial w_2}P_2 + \frac{\partial}{\partial w_3}P_3\right) - \right. \\ &\left.\kappa_*\frac{\partial}{\partial w_1}\left(\frac{\partial^2}{\partial {w_1}^2} + \frac{\partial^2}{\partial {w_2}^2} + \frac{\partial^2}{\partial {w_3}^2}\right)M\right)(w_1, w_2, w_3). \\ &= (E^e f_1)(w_1, w_2, -w_3). \\ &= (E^e f_1)(w_1, w_2, w_3). \\ &\left(\lambda O_2 - \mu_*\left(\frac{\partial^2}{\partial {w_1}^2} + \frac{\partial^2}{\partial {w_2}^2} + \frac{\partial^2}{\partial {w_3}^2}\right)O_2 - v_*\frac{\partial}{\partial w_2}\left(\frac{\partial}{\partial w_1}O_1 + \frac{\partial}{\partial w_2}O_2 + \frac{\partial}{\partial w_3}O_3\right) - \right. \\ &\left.\kappa_*\frac{\partial}{\partial w_2}\left(\frac{\partial^2}{\partial {w_1}^2} + \frac{\partial^2}{\partial {w_2}^2} + \frac{\partial^2}{\partial {w_3}^2}\right)N\right)(w_1, w_2, w_3) \\ &= \left(\lambda P_2 - \mu_*\left(\frac{\partial^2}{\partial {w_1}^2} + \frac{\partial^2}{\partial {w_2}^2} + \frac{\partial^2}{\partial {w_3}^2}\right)P_2 - v_*\frac{\partial}{\partial w_2}\left(\frac{\partial}{\partial w_1}P_1 + \frac{\partial}{\partial w_2}P_2 + \frac{\partial}{\partial w_3}P_3\right) - \right. \\ &\left.\kappa_*\frac{\partial}{\partial w_2}\left(\frac{\partial^2}{\partial {w_1}^2} + \frac{\partial^2}{\partial {w_2}^2} + \frac{\partial^2}{\partial {w_3}^2}\right)M\right)(w_1, w_2, w_3). \end{aligned} \tag{11}$$

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$$= (E^e f_2)(w_1, w_2, -w_3).$$
$$= (E^e f_2)(w_1, w_2, w_3).$$
$$\left(\lambda O_3 - \mu_* \left(\frac{\partial^2}{\partial {w_1}^2} + \frac{\partial^2}{\partial {w_2}^2} + \frac{\partial^2}{\partial {w_3}^2}\right) O_3 - v_* \frac{\partial}{\partial w_3}\left(\frac{\partial}{\partial w_1} O_1 + \frac{\partial}{\partial w_2} O_2 + \frac{\partial}{\partial w_3} O_3\right) - \kappa_* \frac{\partial}{\partial w_3}\left(\frac{\partial^2}{\partial {w_1}^2} + \frac{\partial^2}{\partial {w_2}^2} + \frac{\partial^2}{\partial {w_3}^2}\right) N\right)(w_1, w_2, w_3)$$
$$= \left(\lambda P_3 - \mu_* \left(\frac{\partial^2}{\partial {w_1}^2} + \frac{\partial^2}{\partial {w_2}^2} + \frac{\partial^2}{\partial {w_3}^2}\right) P_3 - v_* \frac{\partial}{\partial w_3}\left(\frac{\partial}{\partial w_1} P_1 + \frac{\partial}{\partial w_2} P_2 + \frac{\partial}{\partial w_3} P_3\right) - \kappa_* \frac{\partial}{\partial w_3}\left(\frac{\partial^2}{\partial {w_1}^2} + \frac{\partial^2}{\partial {w_2}^2} + \frac{\partial^2}{\partial {w_3}^2}\right) M\right)(w_1, w_2, w_3). \quad (3)$$
$$= -(E^0 f_3)(w_1, w_2, -w_3).$$
$$= (E^0 f_3)(w_1, w_2, w_3).$$

Based on the uniqueness of the solution operator in $\mathbf{R}^3$, we obtain:

$$\left.\begin{aligned} O_1(w_1, w_2, w_3) &= P_1(w_1, w_2, w_3) \\ O_2(w_1, w_2, w_3) &= P_2(w_1, w_2, w_3) \\ O_3(w_1, w_2, w_3) &= P_3(w_1, w_2, w_3) \end{aligned}\right\} \quad (13)$$

As a result, from equations (9) and (13), we have $P_3(w_1, w_2, w_3) = -P_3(w_1, w_2, w_3)$. This implies that when $w_3 = 0$, we have $P_3(w_1, w_2, 0) = -P_3(w_1, w_2, 0)$ which is true if and only if $P_3(w_1, w_2, 0) = 0$.

Let ρ and $\mathbf{u}$ be defined as follows:

$$\rho = M + \tilde{\rho} \text{ and } \mathbf{u} = (u_1, u_2, u_3) = (P_1 + \tilde{u}_1, P_2 + \tilde{u}_2, P_3 + \tilde{u}_3) = \mathbf{P} + \tilde{\mathbf{u}} \quad (14)$$

Substitute equation (14) into equation (3), to obtain the following homogeneous system:

$$\lambda\tilde{\rho} + \left(\frac{\partial}{\partial w_1}\tilde{u}_1 + \frac{\partial}{\partial w_2}\tilde{u}_2 + \frac{\partial}{\partial w_3}\tilde{u}_3\right) = 0 \qquad \text{in } \mathbf{R}^3_+$$

$$\lambda\tilde{u}_1 - \mu_* \left(\frac{\partial^2}{\partial {w_1}^2} + \frac{\partial^2}{\partial {w_2}^2} + \frac{\partial^2}{\partial {w_3}^2}\right)\tilde{u}_1 - v_* \frac{\partial}{\partial w_1}\left(\frac{\partial}{\partial w_1}\tilde{u}_1 + \frac{\partial}{\partial w_2}\tilde{u}_2 + \frac{\partial}{\partial w_3}\tilde{u}_3\right) - \kappa_* \frac{\partial}{\partial w_1}\left(\frac{\partial^2}{\partial {w_1}^2} + \frac{\partial^2}{\partial {w_2}^2} + \frac{\partial^2}{\partial {w_3}^2}\right)\tilde{\rho} = 0.$$
$$\lambda\tilde{u}_2 - \mu_* \left(\frac{\partial^2}{\partial {w_1}^2} + \frac{\partial^2}{\partial {w_2}^2} + \frac{\partial^2}{\partial {w_3}^2}\right)\tilde{u}_2 - v_* \frac{\partial}{\partial w_2}\left(\frac{\partial}{\partial w_1}\tilde{u}_1 + \frac{\partial}{\partial w_2}\tilde{u}_2 + \frac{\partial}{\partial w_3}\tilde{u}_3\right) - \kappa_* \frac{\partial}{\partial w_2}\left(\frac{\partial^2}{\partial {w_1}^2} + \frac{\partial^2}{\partial {w_2}^2} + \frac{\partial^2}{\partial {w_3}^2}\right)\tilde{\rho} = 0. \qquad \text{in } \mathbf{R}^3_+ \quad (15)$$
$$\lambda\tilde{u}_3 - \mu_* \left(\frac{\partial^2}{\partial {w_1}^2} + \frac{\partial^2}{\partial {w_2}^2} + \frac{\partial^2}{\partial {w_3}^2}\right)\tilde{u}_3 - v_* \frac{\partial}{\partial w_3}\left(\frac{\partial}{\partial w_1}\tilde{u}_1 + \frac{\partial}{\partial w_2}\tilde{u}_2 + \frac{\partial}{\partial w_3}\tilde{u}_3\right) - \kappa_* \frac{\partial}{\partial w_3}\left(\frac{\partial^2}{\partial {w_1}^2} + \frac{\partial^2}{\partial {w_2}^2} + \frac{\partial^2}{\partial {w_3}^2}\right)\tilde{\rho} = 0.$$

$$\begin{pmatrix} \frac{\partial}{\partial w_1} \\ \frac{\partial}{\partial w_2} \\ \frac{\partial}{\partial w_3} \end{pmatrix} \tilde{\rho} = -\tilde{g}, \text{ where } \tilde{g} = g + \frac{\partial}{\partial w_3}\mathfrak{A}^1(\lambda)\mathcal{K}_\lambda^0\left(E^e\tilde{d}, \mathbf{Ef}\right) \qquad \text{on } \mathbf{R}_0^3$$

$$\begin{pmatrix} \tilde{u}_1 \\ \tilde{u}_2 \end{pmatrix} = \begin{pmatrix} \tilde{h}_1 \\ \tilde{h}_2 \end{pmatrix}, \text{ where } \tilde{h}_j = -\left(\mathfrak{B}^1(\lambda)\mathcal{K}_\lambda^0\left(E^e\tilde{d}, \mathbf{Ef}\right)\right) \text{ for } j = 1,2$$

$$\tilde{u}_3 = 0$$

Consequently, we obtain the homogeneous system of equations from the system (3), which can be conscisely represented as follows:

$$\lambda\rho + \left(\frac{\partial}{\partial w_1}u_1 + \frac{\partial}{\partial w_2}u_2 + \frac{\partial}{\partial w_3}u_1\right) = 0 \qquad \text{in } \mathbf{R}_+^3$$

$$\lambda u_1 - \mu_*\left(\frac{\partial^2}{\partial {w_1}^2} + \frac{\partial^2}{\partial {w_2}^2} + \frac{\partial^2}{\partial {w_3}^2}\right)u_1 - \nu_*\frac{\partial}{\partial w_1}\left(\frac{\partial}{\partial w_1}u_1 + \frac{\partial}{\partial w_2}u_2 + \frac{\partial}{\partial w_3}u_3\right) - \kappa_*\frac{\partial}{\partial w_1}\left(\frac{\partial^2}{\partial {w_1}^2} + \frac{\partial^2}{\partial {w_2}^2} + \frac{\partial^2}{\partial {w_3}^2}\right)\tilde{\rho} = 0.$$

$$\lambda u_2 - \mu_*\left(\frac{\partial^2}{\partial {w_1}^2} + \frac{\partial^2}{\partial {w_2}^2} + \frac{\partial^2}{\partial {w_3}^2}\right)u_2 - \nu_*\frac{\partial}{\partial w_2}\left(\frac{\partial}{\partial w_1}u_1 + \frac{\partial}{\partial w_2}u_2 + \frac{\partial}{\partial w_3}u_3\right) - \kappa_*\frac{\partial}{\partial w_2}\left(\frac{\partial^2}{\partial {w_1}^2} + \frac{\partial^2}{\partial {w_2}^2} + \frac{\partial^2}{\partial {w_3}^2}\right)\rho = 0. \qquad \text{in } \mathbf{R}_+^3$$

$$\lambda u_3 - \mu_*\left(\frac{\partial^2}{\partial {w_1}^2} + \frac{\partial^2}{\partial {w_2}^2} + \frac{\partial^2}{\partial t^2}\right)u_3 - \nu_*\frac{\partial}{\partial w_3}\left(\frac{\partial}{\partial w_1}u_1 + \frac{\partial}{\partial w_2}u_2 + \frac{\partial}{\partial w_3}u_3\right) - \kappa_*\frac{\partial}{\partial w_3}\left(\frac{\partial^2}{\partial {w_1}^2} + \frac{\partial^2}{\partial {w_2}^2} + \frac{\partial^2}{\partial {w_3}^2}\right)\rho = 0. \qquad (16)$$

$$\begin{pmatrix} \frac{\partial}{\partial w_1} \\ \frac{\partial}{\partial w_2} \\ \frac{\partial}{\partial w_3} \end{pmatrix} \rho = -g, \qquad \text{on } \mathbf{R}_0^3$$

$$\begin{pmatrix} u_1 \\ u_2 \end{pmatrix} = \begin{pmatrix} h_1 \\ h_2 \end{pmatrix},$$

$$u_3 = 0$$

Solving the Homogeneous System in $\mathbf{R}_+^3$

The next step in proving Theorem 2 involves solving the homogeneous system of equations (16) by establishing Theorem 3, as outlined below.

$$\mathcal{G}_\lambda \boldsymbol{G} = \left(\begin{pmatrix} \frac{\partial^2}{\partial {w_1}^2}(g) & \frac{\partial^2}{\partial w_1 \partial w_2}(g) & \frac{\partial^2}{\partial w_1 \partial w_3}(g) \\ \frac{\partial^2}{\partial w_1 \partial w_2}(g) & \frac{\partial^2}{\partial {w_2}^2}(g) & \frac{\partial^2}{\partial w_2 \partial w_3}(g) \\ \frac{\partial^2}{\partial w_1 \partial w_3}(g) & \frac{\partial^2}{\partial w_2 \partial w_3}(g) & \frac{\partial^2}{\partial {w_3}^2}(g) \end{pmatrix}, \lambda^{\frac{1}{2}} \frac{\partial}{\partial w_1} g, \lambda^{\frac{1}{2}} \frac{\square\square}{\partial w_2} g, \lambda^{\frac{1}{2}} \frac{\partial}{\partial w_3} g, \lambda g, \right.$$

$$\begin{pmatrix} \frac{\partial^2}{\partial {w_1}^2}(h_1) & \frac{\partial^2}{\partial w_1 \partial w_2}(h_1) & \frac{\partial^2}{\partial w_1 \partial w_3}(h_1) \\ \frac{\partial^2}{\partial w_1 \partial w_2}(h_1) & \frac{\partial^2}{\partial {w_2}^2}(h_1) & \frac{\partial^2}{\partial w_2 \partial w_3}(h_1) \\ \frac{\partial^2}{\partial w_1 \partial w_3}(h_1) & \frac{\partial^2}{\partial w_2 \partial w_3}(h_1) & \frac{\partial^2}{\partial {w_3}^2}(h_1) \end{pmatrix}, \lambda^{\frac{1}{2}} \frac{\partial}{\partial w_1}(h_1), \lambda^{\frac{1}{2}} \frac{\partial}{\partial w_2}(h_1), \lambda^{\frac{1}{2}} \frac{\partial}{\partial w_3}(h_1), \lambda h_1,$$

$$\left. \begin{pmatrix} \frac{\partial^2}{\partial {w_1}^2}(h_2) & \frac{\partial^2}{\partial w_1 \partial w_2}(h_2) & \frac{\partial^2}{\partial w_1 \partial w_3}(h_2) \\ \frac{\partial^2}{\partial w_1 \partial w_2}(h_2) & \frac{\partial^2}{\partial {w_2}^2}(h_2) & \frac{\partial^2}{\partial w_2 \partial w_3}(h_2) \\ \frac{\partial^2}{\partial w_1 \partial w_3}(h_2) & \frac{\partial^2}{\partial w_2 \partial w_3}(h_2) & \frac{\partial^2}{\partial {w_3}^2}(h_2) \end{pmatrix}, \lambda^{\frac{1}{2}} \frac{\partial}{\partial w_1}(h_2), \lambda^{\frac{1}{2}} \frac{\partial}{\partial w_2}(h_2), \lambda^{\frac{1}{2}} \frac{\partial}{\partial w_3}(h_2), \lambda h_2 \right)$$

$$\in Y_p^2(\mathbf{R}_+^3).$$

For the right-hand side function $\boldsymbol{G} = (g, h_1, h_2)$ in equation (14), the space are

$$\mathcal{Y}_p(\mathbf{R}_+^3) = W_p^2(\mathbf{R}_+^3)^3, \mathfrak{Y}_p(\mathbf{R}_+^3) = L_p(\mathbf{R}_+^3)^C.$$

Then, defined as follows:

$$Y_p^2(\mathbf{R}_+^3) = L_p(\mathbf{R}_+^3)^C,$$

$$C = 3(3^2 + 3 + 1) = 39,$$

Theorem 3. Let $p \in (1, \infty)$ and assume that μ_*, ν_* dan κ_* are positive constants that satisfy $\left(\frac{\mu_* + \nu_*}{2\kappa_*}\right)^2 - \left(\frac{1}{\kappa_*}\right) < 0$ and $\left(\frac{\mu_* + \nu_*}{2\kappa_*}\right)^2 - \left(\frac{1}{\kappa_*}\right) > 0, \kappa_* \neq \mu_* \nu_*$. Then, for any $\lambda \in \boldsymbol{C}_+$ there exist operators $\mathcal{A}^2(\lambda)$ and $\mathcal{B}^2(\lambda)$ with,

$$\mathfrak{A}^2(\lambda) \in Hol\left(\boldsymbol{C}_+, \mathcal{L}\left(Y_q^2(\boldsymbol{R}_+^3), W_q^3(\boldsymbol{R}_+^3)\right)\right),$$

$$\mathfrak{B}^2(\lambda) \in Hol\left(\boldsymbol{C}_+, \mathcal{L}\left(Y_q^2(\boldsymbol{R}_+^3), W_q^2(\boldsymbol{R}_+^3)^3\right)\right),$$

such that, for any $\boldsymbol{G} = (g, h_1, h_2) \in \mathcal{Y}_P(\boldsymbol{R}_+^3)$ a unique solution to the system (16) is $(\rho, \boldsymbol{u}) = (\mathfrak{A}^2(\lambda)\mathcal{G}_\lambda \boldsymbol{G}, \mathfrak{B}^2(\lambda)\, \mathcal{G}_\lambda \boldsymbol{G})$.

Proof

Before proving Theorem 3, we introduce the partial Fourier transform. For a function $u = u(w_1, w_2, w_3)$ defined on $\mathbf{R}^3$, the partial Fourier transform and its inverse are given by:

$$\hat{u} = \hat{u}(w_3) = \hat{u}(\xi_1, \xi_2, w_3) = \int_{\mathbf{R}} \int_{\mathbf{R}} e^{-i(w_1,w_2)\cdot(\xi_1,\xi_2)} u(w_1, w_2, w_3)\, dw_1 dw_2$$

$$\mathcal{F}^{-1}_{(\xi_1,\xi_2)}[\hat{u}(\xi_1, \xi_2, w_3)](w_1, w_2) = \frac{1}{(2\pi)^2} \int_{\mathbf{R}} \int_{\mathbf{R}} e^{-i(w_1,w_2)\cdot(\xi_1,\xi_2)} u(w_1, w_2, w_3)\, dw_1 dw_2$$

Let $\phi = \left(\frac{\partial}{\partial w_1} u_1 + \frac{\partial}{\partial w_2} u_2 + \frac{\partial}{\partial w_3} u_3\right)$. The first equation of (14) becomes $\lambda\hat{\rho} + \hat{\phi} = 0$. Applying the partial Fourier transform to the system (14) gives the following ordinary differential equations:

$$\lambda\hat{\rho} + \hat{\phi} = 0, \qquad w_3 > 0 \tag{4}$$

$$\lambda\hat{u}_j - \mu_*\left(\frac{\partial^2}{\partial {w_3}^2} - |\xi_1^2 + \xi_2^2|\right)\hat{u}_j - v_* i\xi_j\hat{\phi} - \kappa_* i\xi_j\left(\frac{\partial^2}{\partial {w_3}^2} - |\xi_1^2 + \xi_2^2|\right)\hat{\rho} = 0, \quad w_3 > 0 \tag{18}$$

$$\lambda\hat{u}_3 - \mu_*\left(\frac{\partial^2}{\partial {w_3}^2} - |\xi_1^2 + \xi_2^2|\right)\hat{u}_3 - v_*\frac{\partial}{dw_3}\hat{\phi} - \kappa_*\frac{\partial}{dw_3}\left(\frac{\partial^2}{\partial {w_3}^2} - |\xi_1^2 + \xi_2^2|\right)\hat{\rho} = 0, \quad w_3 > 0 \tag{19}$$

with boundary conditions:

$$\frac{\partial}{dw_3}\hat{\rho}(0) = -\hat{g}(0). \tag{20}$$

$$\begin{pmatrix} \hat{u}_j(0) \\ \hat{u}_3(0) \end{pmatrix} = \begin{pmatrix} \hat{h}_j(0) \\ (0) \end{pmatrix}. \tag{21}$$

where,

$$\hat{\phi} = \sum_{j=1}^{2} i\xi_j\hat{u}_j + \frac{\partial}{dw_3}\hat{u}_3. \tag{5}$$

We define a polynomial $\boldsymbol{P_\lambda(k)}$ as follows,

$$P_\lambda(k) = \lambda^2 - \lambda(\mu_* + v_*)\kappa_*(k^2 - |\xi_1^2 + \xi_2^2|)^2 + (k^2 - |\xi_1^2 + \xi_2^2|) \tag{6}$$

Equation (17) can be rewritten as.

$$\hat{\rho} = -\frac{\hat{\phi}}{\lambda}. \tag{7}$$

Substitute equation (21) into equation (18) dan (19) gives:

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$$\lambda^2 \hat{u}_j - \lambda\mu_* \left(\frac{\partial^2}{\partial {w_3}^2} - |\xi_1^2 + \xi_2^2|\right) \hat{u}_j - i\xi_j \left(\lambda v_* - \kappa_* \left(\frac{\partial^2}{\partial {w_3}^2} - |\xi_1^2 + \xi_2^2|\right)\right) \hat{\phi} = 0 \quad , \quad w_3 > 0 \tag{25}$$

$$\lambda^2 \hat{u}_3 - \lambda\mu_* \left(\frac{\partial^2}{\partial {w_3}^2} - |\xi_1^2 + \xi_2^2|\right) \hat{u}_3 - \frac{\partial}{dw_3} \left(\lambda v_* - \kappa_* \left(\frac{\partial^2}{\partial {w_3}^2} - |\xi_1^2 + \xi_2^2|\right)\right) \hat{\phi} = 0 \quad , \quad w_3 > 0 \tag{26}$$

and to equation (20), we obtain:

$$\frac{\partial}{dw_3} \hat{\phi}(0) = \lambda \hat{g}(0). \tag{8}$$

this leads to the following ordinary differential system,

$$P_\lambda \left(\frac{\partial}{dw_3}\right) \hat{\phi} = 0, \qquad w_3 > 0 \tag{28}$$

$$\left(\frac{\partial^2}{\partial {w_3}^2} - \omega_\lambda^2\right) P_\lambda \left(\frac{\partial}{dw_3}\right) \hat{u}_J = 0, \qquad w_3 > 0, untuk\ J = 1,2,3 \tag{29}$$

where $\omega_\lambda = \sqrt{|\xi_1^2 + \xi_2^2| + \frac{\lambda}{\mu_*}}$ so that $\omega_\lambda^2 = |\xi_1^2 + \xi_2^2| + \frac{\lambda}{\mu_*}$. Next, substituting ω_λ ke into equations (24) and (25) yields:

$$\lambda\mu \left(\frac{\partial^2}{\partial {w_3}^2} - \omega_\lambda^2\right) \hat{u}_j + i\xi_j \left(\lambda v_* - \kappa_* \left(\frac{\partial^2}{\partial {w_3}^2} - |\xi_1^2 + \xi_2^2|\right)\right) \hat{\phi} = 0,\ w_3 > 0 \tag{30}$$

$$\lambda\mu \left(\frac{\partial^2}{\partial {w_3}^2} - \omega_\lambda^2\right) \hat{u}_3 + \frac{\partial}{dw_3} \left(\lambda v_* - \kappa_* \left(\frac{\partial^2}{\partial {w_3}^2} - |\xi_1^2 + \xi_2^2|\right)\right) \hat{\phi} = 0,\ w_3 > 0 \tag{31}$$

We then define $d = \left(\frac{k^2 - |\xi_1^2 + \xi_2^2|}{\lambda}\right)$, so equation (23) can be rewritten as:

$$P_\lambda(k) = \kappa \lambda^2 p(d). \tag{9}$$

Thus, the polynomial $p(d)$ is defined as follows:

$$p(d) = d^2 - \frac{\mu_* + v_*}{\kappa_*} d + \frac{1}{\kappa_*},$$

which yields the roots of $p(d)$ as:

$$d_n = \begin{cases} \dfrac{\mu_* + v_*}{2\kappa_*} \pm \sqrt{\eta}\,, & (\eta \geq 0) \\ \dfrac{\mu_* + v_*}{2\kappa_*} \pm i\sqrt{|\eta|}, & (\eta < 0) \end{cases}$$

with $i = \sqrt{-1}$ and $\eta = \left(\frac{\mu_* + v_*}{2\kappa_*}\right)^2 - \frac{1}{\kappa_*}, \eta \neq 0$. These can also be written as $d_n, n = 1,2,3,4$ where:

$d_1 = \frac{\mu_* + v_*}{2\kappa_*} + \sqrt{\eta}$ dan $\quad d_2 = \frac{\mu_* + v_*}{2\kappa_*} - \sqrt{\eta} \quad$ untuk $\eta \geq 0$

$d_3 = \frac{\mu_* + v_*}{2\kappa_*} + i\sqrt{|\eta|}$ dan $\quad d_4 = \frac{\mu_* + v_*}{2\kappa_*} - i\sqrt{|\eta|} \quad$ untuk $\eta < 0$

considering $d_n = \left(\frac{k^2 - |\xi_1^2 + \xi_2^2|}{\lambda}\right), n = 1,2,3,4$ the roots of equation (32) are as follows:

$$k_{1n} = -\sqrt{|\xi_1^2 + \xi_2^2| + \lambda d_n} \quad \text{dan} \quad k_{2n} = \sqrt{|\xi_1^2 + \xi_2^2| + \lambda d_n}.$$

Thus, k_{2n} $(n = 1,2,3,4)$ are the characteristic roots of equations (28) and (29). Further characteristic roots from equation (31) are then obtained:

$k_1 = -\omega_\lambda$ dan $k_2 = \omega_\lambda$

given $\omega_\lambda = \sqrt{|\xi_1^2 + \xi_2^2| + \frac{\lambda}{\mu_*}}$. The characteristics roots k_{2n} $(n = 1,2,3,4)$ and k_2 are the roots of equation (29). Therefore, the general solutions for equations (28) and (29) are:

$$\hat{\phi} = \sigma e^{-k_{21} w_3} + \tau e^{-k_{22} w_3} \tag{10}$$

$$\hat{u}_J = \alpha_J e^{-\omega_\lambda w_3} + \beta_J (e^{-k_{21} w_3} - e^{-\omega_\lambda w_3}) + \gamma_J (e^{-k_{22} w_3} - e^{-\omega_\lambda w_3}), \qquad J = 1,2,3 \tag{11}$$

$$\frac{\partial}{dw_3} \hat{u}_J = (-\omega_\lambda \alpha_J + \omega_\lambda \beta_J + \omega_\lambda \gamma_J) e^{-\omega_\lambda w_3} - k_{21} \beta_J e^{-k_{21} w_3} - k_{22} \gamma_J e^{-k_{22} w_3}, \ J = 1,2,3 \tag{12}$$

From equation (22), we obtain:

$$\sigma = i\xi' \cdot \beta' - k_{21} \beta_3, \ \tau = i\xi' \cdot \gamma' - k_{22} \gamma_3 \tag{13}$$

$$i\xi' \cdot \alpha' - i\xi' \cdot \beta' - i\xi' \cdot \gamma' - \omega_\lambda \alpha_3 + \omega_\lambda \beta_3 + \omega_\lambda \gamma_3 = 0, \tag{14}$$

where $i\xi' \cdot x' = \sum_{j=1}^{2} i\xi_j a_j$ for $x \in \{\alpha', \beta', \gamma'\}$.

Substitute $\kappa_* \neq \mu_* v_*$ into equation (33) and (34) and then into equations (30) and (31):

$$\lambda \mu_* \beta_j (k_{21}^2 - \omega_\lambda^2) + i\xi_j (i\xi' \cdot \beta' - k_{21} \beta_3)(v_* \lambda - \kappa_* (k_{21}^2 - |\xi_1^2 + \xi_2^2|)) = 0,$$

$$\lambda \mu_* \gamma_j (k_{22}^2 - \omega_\lambda^2) + i\xi_j (i\xi' \cdot \gamma' - k_{22} \gamma_3)(v_* \lambda - \kappa_* (k_{22}^2 - |\xi_1^2 + \xi_2^2|)) = 0,$$

$$\lambda \mu_* \beta_3 (k_{21}^2 - \omega_\lambda^2) - k_{21} (i\xi' \cdot \beta' - k_{21} \beta_3)(v_* \lambda - \kappa_* (k_{21}^2 - |\xi_1^2 + \xi_2^2|)) = 0,$$

$$\lambda \mu_* \gamma_3 (k_{22}^2 - \omega_\lambda^2) - k_{22} (i\xi' \cdot \gamma' - k_{22} \gamma_3)(v_* \lambda - \kappa_* (k_{22}^2 - |\xi_1^2 + \xi_2^2|)) = 0,$$

as a result,

$$(k_{21}^2 - \omega_\lambda^2)\left(\beta_j + \frac{i\xi_j}{k_{21}} \beta_3\right) = 0, \qquad j = 1,2$$

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$$(k_{22}^2-\omega_\lambda^2)\left(\gamma_j+\frac{i\xi_j}{k_{22}}\gamma_3\right)=0,\qquad j=1,2$$

since $\omega_\lambda \neq k_{21}$ and $\omega_\lambda \neq k_{22}$, the coefficient β_j and γ_j are given by,

$$\beta_j=-\frac{i\xi_j}{k_{21}}\beta_3,\;\; \gamma_j=-\frac{i\xi_j}{k_{22}}\gamma_3,\; j=1,2. \tag{15}$$

Next, multiplying the coefficients β_j and γ_j by $i\xi_j$ yields:

$$i\xi'\cdot\beta'=\frac{|\xi_1^2+\xi_2^2|}{k_{21}}\beta_3,\qquad i\xi'\cdot\gamma'=\frac{|\xi_1^2+\xi_2^2|}{k_{22}}\gamma_3 \tag{16}$$

as a result,

$$i\xi'\cdot\beta'-k_{21}\beta_3=-\left(\frac{k_{21}^2-|\xi_1^2+\xi_2^2|}{k_{21}}\right)\beta_3,\;\; i\xi'\cdot\gamma'-k_{22}\gamma_3=-\left(\frac{k_{21}^2-|\xi_1^2+\xi_2^2|}{k_{22}}\right)\gamma_3. \tag{40}$$

To determine the coefficients α_j and α_3, subtitute equation (34) into the boundary condition (21), resulting in

$$\alpha_j=\hat{h}_j(0), \tag{41}$$

$$\alpha_3=0. \tag{17}$$

Then, multiplying equation (41) by $i\xi_j, j=1,2$, yields:

$$i\xi'\cdot\alpha'=i\xi'\cdot\hat{\mathbf{h}}'(0),\quad \hat{\mathbf{h}}'(0)=\left(\hat{h}_1(0),\hat{h}_2(0)\right)^{\intercal}. \tag{18}$$

Substitute equation (33) into boundary condition (26), provides

$$-k_{21}\sigma-k_{22}\tau=\lambda\hat{g}(0). \tag{19}$$

Next, substitute equation (40) into equation (44) to obtain:

$$\lambda\hat{g}(0)\quad=(k_{21}^2-|\xi_1^2+\xi_2^2|)\beta_3+(k_{22}^2-|\xi_1^2+\xi_2^2|)\gamma_3. \tag{20}$$

Then, perform the simultaneous differentiation related to β_3 and γ_3 by substituting equations (39), (41), and (45) into equation (37) resulting in:

$$i\xi'\cdot\hat{\mathbf{h}}'(0)-\frac{|\xi_1^2+\xi_2^2|}{k_{21}}\beta_3-\frac{|\xi_1^2+\xi_2^2|}{k_{22}}\gamma_3+\omega_\lambda\beta_3+\omega_\lambda\gamma_3=0 \tag{46}$$

Therefore,

$$k_{21}k_{22}i\xi'\cdot\hat{\mathbf{h}}'(0)=-k_{22}(k_{21}\omega_\lambda-|\xi_1^2+\xi_2^2|)\beta_3-k_{21}(k_{22}\omega_\lambda-|\xi_1^2+\xi_2^2|)\gamma_3 \tag{21}$$

Based on equations (45) and (47), we can form the following matrix:

$$\mathbf{L}\begin{pmatrix}\beta_3\\ \gamma_3\end{pmatrix}\;=\begin{pmatrix}\lambda\hat{g}(0)\\ k_{21}k_{22}i\xi'\cdot\hat{\mathbf{h}}'(0)\end{pmatrix} \tag{22}$$

$$\mathbf{L}\;=\begin{pmatrix}(k_{21}^2-|\xi_1^2+\xi_2^2|) & (k_{22}^2-|\xi_1^2+\xi_2^2|)\\ -k_{22}(k_{21}\omega_\lambda-|\xi_1^2+\xi_2^2|) & -k_{21}(k_{22}\omega_\lambda-|\xi_1^2+\xi_2^2|)\end{pmatrix} \tag{23}$$

To solve equations (48) and (49), calculate the determinant of $\mathbf{L}$:

$$\det \mathrm{L}\;=k_{22}(k_{22}^2-|\xi_1^2+\xi_2^2|)(k_{21}\omega_\lambda-|\xi_1^2+\xi_2^2|)-k_{21}(k_{21}^2-|\xi_1^2+\xi_2^2|)(k_{22}\omega_\lambda-|\xi_1^2+\xi_2^2|)$$

$$= (k_{22} - k_{21})\big((k_{22} + k_{21})(k_{21}k_{22}\omega_\lambda)\big) - |\xi_1^2 + \xi_2^2|(k_{22}^2 + k_{21}k_{22} + k_{21}^2 - |\xi_1^2 + \xi_2^2|). \quad (50)$$

Lemma 4 There holds $\det \mathbf{L} \neq 0$ for any $(\xi', \lambda) \in \mathbf{R}^2 \times (\overline{\mathbf{C}_+}\backslash\{0\})$, where $\overline{\mathbf{C}_+} = \{z \in \mathbf{C} | \Re_Z \geq 0\}$.

Proof. The proof is by contradiction. Assume $\det \mathbf{L} = 0$ for some $(\xi', \lambda) \in \mathbf{R}^2 \times (\overline{\mathbf{C}_+}\backslash\{0\})$. Then, there exists a non-trivial solution $(\beta_3, \gamma_3) \neq (0,0)$ that satisfies equation (48) with $\hat{g}(0) = 0$ and $\hat{h}_1(0) = 0 \text{ dan } \hat{h}_2(0) = 0$. This would imply that equations (29) and (30), with $\hat{\phi} = \sum_{j=1}^{2} i\xi_j\, \hat{u}_j + \frac{\partial}{dw_3}\hat{u}_3$ and homogeneous boundary conditions $\hat{u}_J(0) = 0$ and $\frac{\partial}{dw_3}\hat{\phi}(0) = 0$, admit a non-trivial, smooth solution $(\hat{\mathrm{u}}_1, \hat{\mathrm{u}}_2, \hat{\mathrm{u}}_3, \hat{\phi})$ that decays exponentially as $w_3 \to \infty$.

Recall that equations (29) and (30) are respectively equivalent to equations (24) and (25). By multiplying both (24) and (25) by λ^{-1} , we obtain the following:

$$\lambda\hat{u}_j - \mu_*\left(\frac{\partial^2}{\partial {w_3}^2} - |\xi_1^2 + \xi_2^2|\right)\hat{u}_j - i\xi_j\left(\lambda v_* - \kappa_*\lambda^{-1}\left(\frac{\partial^2}{\partial {w_3}^2} - |\xi_1^2 + \xi_2^2|\right)\right)\hat{\phi} = 0, w_3 > 0, \quad (51)$$

$$\lambda\hat{u}_3 - \mu_*\left(\frac{\partial^2}{\partial {w_3}^2} - |\xi_1^2 + \xi_2^2|\right)\hat{u}_3 - \frac{\partial}{\partial w_3}\left(\lambda v_* - \kappa_*\lambda^{-1}\left(\frac{\partial^2}{\partial {w_3}^2} - |\xi_1^2 + \xi_2^2|\right)\right)\hat{\phi} = 0, w_3 > 0 \quad (24)$$

In this proof, we desine the inner product $(a, b) = \int_0^\infty a(w_3)\overline{b(w_3)}\frac{\partial}{\partial w_3}$ and the norm $\|a\| = \sqrt{(a, a)}$ for functions $a = a(w_3)$ and $b = b(w_3)$ on $\mathbf{R}_+$.

Step 1. By multiplying equation (51) by $\overline{\hat{u}_J(w_3)}$ and integrating over $w_3 \in (0, \infty)$, we obtain:

$$\lambda\|u_j\|^2 - \mu_*\left(\left(\frac{\partial^2}{\partial {w_3}^2}u_j, u_j\right) - |\xi_1^2 + \xi_2^2|\|u_j\|^2\right) - v_*(i\xi_j\phi, u_j) + \kappa_*\lambda^{-1}\left(i\xi_j\left(\frac{\partial^2}{\partial {w_3}^2} - |\xi_1^2 + \xi_2^2|\right)\phi, u_j\right) = 0. \quad (25)$$

Using integration by parts, along with the condition n $\mathrm{u_j}(0) = 0$, and the identities

$$\left(\frac{\partial^2}{\partial {w_3}^2}u_j, u_j\right) = -\left\|\frac{\partial^2}{\partial {w_3}^2}u_j\right\|^2, \text{where } j = 1,2$$

$$(i\xi_j\phi, u_j) = -(\phi, i\xi_j, u_j),$$

$$\left(i\xi_j\left(\frac{\partial^2}{\partial {w_3}^2} - |\xi_1^2 + \xi_2^2|\right)\phi, u_j\right) = -\left(\left(\frac{\partial^2}{\partial {w_3}^2} - |\xi_1^2 + \xi_2^2|\right)\phi, i\xi_j u_j\right),$$

this simplifies to:

$$\lambda\|u_j\|^2 + \mu_*\left(\left\|\frac{\partial}{\partial w_3}u_j\right\|^2 + |\xi_1^2 + \xi_2^2|\|u_j\|^2\right) + v_*(\phi, i\xi_j u_j) - \kappa_*\lambda^{-1}\left(\left(\frac{\partial^2}{\partial w_3{}^2} - |\xi_1^2 + \xi_2^2|\right)\phi, i\xi_j u_j\right) = 0. \quad (26)$$

Similarly, from equation (52), we derive:

$$\lambda\|u_3\|^2 + \mu_*\left(\left\|\frac{\partial}{\partial w_3}u_3\right\|^2 + |\xi_1^2 + \xi_2^2|\|u_3\|^2\right) + v_*\left(\phi, \frac{\partial}{\partial w_3}u_3\right) - \kappa_*\lambda^{-1}\left(\left(\frac{\partial^2}{\partial w_3{}^2} - |\xi_1^2 + \xi_2^2|\right)\phi, \frac{\partial}{\partial w_3}u_3\right) = 0. \quad (27)$$

Step 2. Add equations (55) and (56), with $\phi = \sum_{j=1}^{2} i\xi_j\,\hat{u}_j + \frac{\partial}{dw_3}\hat{u}_3$ to obtain,

$$\lambda\sum_{J=1}^{3}\|u_J\|^2 + \mu_*\sum_{J=1}^{3}\left(\left\|\frac{\partial}{\partial w_3}u_J\right\|^2 + |\xi_1^2 + \xi_2^2|\|u_J\|^2\right) + v_*\|\phi\|^2 - \kappa_*\lambda^{-1}\left(\left(\frac{\partial^2}{\partial w_3{}^2} - |\xi_1^2 + \xi_2^2|\right)\phi, \phi\right) = 0. \quad (28)$$

Next, by applying integration by parts to the last term in equation (55) and using the condition $\frac{\partial}{dw_3}\phi(0) = 0$, we get:

$$\left(\left(\frac{\partial^2}{\partial w_3{}^2} - |\xi_1^2 + \xi_2^2|\right)\phi, \phi\right) = -\left\|\frac{\partial}{\partial w_3}u_j\right\|^2 - |\xi_1^2 + \xi_2^2|\|\phi\|^2. \quad (29)$$

Substituting equation (56) into equation (57) with $\lambda^{-1} = \bar{\lambda}|\lambda|^{-2}$, we obtain:

$$\lambda\sum_{J=1}^{3}\|u_J\|^2 + \mu_*\sum_{J=1}^{3}\left(\left\|\frac{\partial}{\partial w_3}u_J\right\|^2 + |\xi_1^2 + \xi_2^2|\|u_J\|^2\right) + v_*\|\phi\|^2 + \kappa_*\bar{\lambda}|\lambda|^{-2}\left(\left\|\frac{\partial}{\partial w_3}\phi\right\|^2 + |\xi_1^2 + \xi_2^2| + \|\phi\|^2\right) = 0. \quad (30)$$

Step 3. Separate the real and imaginary parts of equation (59) to obtain:

$$(\Re\lambda)\left\{\sum_{J=1}^{3}\|\mathrm{u}_J\|^2 + \kappa_*|\lambda|^{-2}\left(\left\|\frac{\partial}{\partial w_3}\phi\right\|^2 + |\xi_1^2 + \xi_2^2|\|\phi\|^2\right)\right\} + \mu_*\sum_{J=1}^{3}\left(\left\|\frac{\partial}{\partial w_3}\mathrm{u}_J\right\|^2 + |\xi_1^2 + \xi_2^2|\|\mathrm{u}_J\|^2\right) + v_*\|\phi\|^2 = 0, \quad (31)$$

$$(\Im\lambda)\left\{\sum_{J=1}^{3}\|\mathrm{u}_J\|^2 - \kappa_*|\lambda|^{-2}\left(\left\|\frac{\partial}{\partial w_3}\phi\right\|^2 + |\xi_1^2 + \xi_2^2|\|\phi\|^2\right)\right\} = 0. \quad (60)$$

Using equation (57), we can conclude that $\phi = 0$. When $\Re\lambda > 0$, based on equation (57), it follows that $(u_1, u_2, u_3) = (0,0,0)$, and when $\Re\lambda = 0$, from equation (58), it also follows that $(u_1, u_2, u_3) = (0,0,0)$. Therefore, $(\mathrm{u}_1, \mathrm{u}_2, \mathrm{u}_3, \phi) = (0,0,0,0)$, which contradicts the fact that $(\mathrm{u}_1, \mathrm{u}_2, \mathrm{u}_3, \phi)$ is a non-trivial solution. Thus, Lemma 4 is proven.

Given that the inverse of det L is written as $\mathbf{L}^{-1}$, we have:

$$\mathbf{L}^{-1} = \frac{1}{\det \mathbf{L}} \begin{pmatrix} L_{11} & L_{12} \\ L_{21} & L_{22} \end{pmatrix},$$

where,

$$\begin{aligned} L_{11} &= -k_{21}\left(k_{22}\omega_\lambda - \left|\xi_1^2 + \xi_2^2\right|\right), & L_{12} &= -\left(k_{22}^2 - \left|\xi_1^2 + \xi_2^2\right|\right). \\ L_{21} &= k_{22}\left(k_{21}\omega_\lambda - \left|\xi_1^2 + \xi_2^2\right|\right), & L_{22} &= k_{21}^2 - \left|\xi_1^2 + \xi_2^2\right|. \end{aligned} \tag{61}$$

By solving equation (47), the coefficients β_3 and γ_3 are obtained as follows:

$$\begin{aligned} \beta_3 &= \frac{\lambda L_{11}}{\det \mathbf{L}} \hat{g}(0) + \frac{k_{21}k_{22}L_{12}}{\det \mathbf{L}} i\xi' \cdot \widehat{\mathbf{h}'}(0) \\ \gamma_3 &= \frac{\lambda L_{21}}{\det \mathbf{L}} \hat{g}(0) + \frac{k_{21}k_{22}L_{22}}{\det \mathbf{L}} i\xi' \cdot \widehat{\mathbf{h}'}(0) \end{aligned} \tag{32}$$

Substitute the coefficients into the general solution of the characteristic equation. Using equations (33), (34), (36), (38), (39), (40), and (41), we obtain:

$$\hat{\rho}(w_3) = \left(\frac{k_{21}^2 - \left|\xi_1^2 + \xi_2^2\right|}{\lambda k_{21}}\right) e^{-k_{21}w_3}\beta_3 + \left(\frac{k_{22}^2 - \left|\xi_1^2 + \xi_2^2\right|}{\lambda k_{22}}\right) e^{-k_{22}w_3}\gamma_3 \tag{33}$$

$$\hat{u}_j(w_3) = \hat{h}_j(0)e^{-\omega_\lambda w_3} - \frac{i\xi_j}{k_{21}}\beta_3\left(e^{-k_{21}w_3} - e^{-\omega_\lambda w_3}\right) - \frac{i\xi_j}{k_{22}}\gamma_3\left(e^{-k_{22}w_3} - e^{-\omega_\lambda w_3}\right) \tag{34}$$

$$\hat{u}_3(w_3) = \beta_3\left(e^{-k_{21}w_3} - e^{-\omega_\lambda w_3}\right) + \gamma_3\left(e^{-k_{22}w_3} - e^{-\omega_\lambda w_3}\right), \tag{35}$$

Given,

$$\mathcal{M}_0(w_3) = \frac{e^{-k_{22}w_3} - e^{-k_{21}w_3}}{k_{22} - k_{21}}, \qquad \mathcal{M}_i(w_3) = \frac{e^{-k_i w_3} - e^{-\omega_\lambda w_3}}{k_{22} - k_{21}} \qquad (i = 1,2) \tag{36}$$

We know:

$$\omega_\lambda^2 = \left|\xi_1^2 + \xi_2^2\right| + \lambda{\mu_*}^{-1},\ k_i^2 - \left|\xi_1^2 + \xi_2^2\right| = s_i\lambda,\ k_i^2 - \omega_\lambda^2 = (s_i - {\mu_*}^{-1})\lambda,\ (i = 1,2) \tag{37}$$

thus,

$$d_2 - d_1 = \frac{k_{22}^2 - k_{21}^2}{\lambda} = \frac{(k_{22} + k_{21})(k_{22} - k_{21})}{\lambda}. \tag{38}$$

Also define the following symbols:

$$\begin{aligned} \mathfrak{m}_i(\xi_1, \xi_2, \lambda) &= \frac{k_{2i}(k_{2i} + \omega_\lambda)\det \mathbf{L}}{\lambda(k_{22} - k_{21})} \qquad (i = 1,2), \\ \mathfrak{n}_1(\xi_1, \xi_2, \lambda) &= \frac{(k_{22} + \omega_\lambda)L_{11}}{\lambda}, \qquad \mathfrak{n}_2(\xi_1, \xi_2, \lambda) = \frac{(k_{21} + \omega_\lambda)L_{21}}{\lambda}, \\ \mathfrak{p}_1(\xi_1, \xi_2, \lambda) &= \frac{k_{21} + \omega_\lambda}{k_{22} + \omega_\lambda}, \qquad \mathfrak{p}_2(\xi_1, \xi_2, \lambda) = \frac{k_{22} + \omega_\lambda}{k_{21} + \omega_\lambda}. \end{aligned} \tag{39}$$

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Using the coefficients above, β_3 and γ_3 can be expressed as follows:

$$\beta_3 \quad = \frac{k_{21}(k_{21}+\omega_\lambda)L_{11}}{(k_{22}-k_{21})\mathfrak{m}_1(\xi_1,\xi_2,\lambda)}\hat{g}(0) - \frac{d_2k_{21}^2k_{22}(k_{21}+\omega_\lambda)}{(k_{22}-k_{21})\mathfrak{m}_1(\xi_1,\xi_2,\lambda)}\left(i\xi_1\hat{h}_1 + i\xi_2\hat{h}_2\right)(0), \tag{70}$$

$$\gamma_3 \quad = \frac{k_{22}(k_{22}+\omega_\lambda)L_{21}}{(k_{22}-k_{21})\mathfrak{m}_2(\xi_1,\xi_2,\lambda)}\hat{g}(0) - \frac{d_1k_{21}^2k_{21}(k_{22}+\omega_\lambda)}{(k_{22}-k_{21})\mathfrak{m}_2(\xi_1,\xi_2,\lambda)}\left(i\xi_1\hat{h}_1 + i\xi_2\hat{h}_2\right)(0). \tag{71}$$

Therefore, the solution operators for ρ, u_j, and u_3 are obtained by applying the inverse Fourier transform to equations (64), (65), and (66) as follows:

$$\begin{aligned}
\rho \quad &= \sum_{i=1}^{2}\mathcal{F}^{-1}_{(\xi_1\xi_2)}\left[\frac{d_i(k_{22}+k_{21})\mathfrak{p}_i(\xi_1,\xi_2,\lambda)\mathfrak{n}_i(\xi_1,\xi_2,\lambda)}{(d_2-d_1)\mathfrak{m}_i(\xi_1,\xi_2,\lambda)}e^{-k_{21}w_3}\hat{g}(0)\right](w_1,w_2) - \\
&\sum_{l=1}^{2}\mathcal{F}^{-1}_{(\xi_1\xi_2)}\left[\frac{d_1d_2i\xi_lk_{21}(k_{21}+\omega_\lambda)}{\mathfrak{m}_1(\xi_1,\xi_2,\lambda)}e^{-k_{21}w_3}\hat{h}_l(0)\right](w_1,w_2) + \\
&\mathcal{F}^{-1}_{(\xi_1\xi_2)}\left[\frac{d_2(k_{22}+\omega_\lambda)L_{21}}{\mathfrak{m}_2(\xi_1,\xi_2,\lambda)}\mathcal{M}_0(w_3)\hat{g}(0)\right](w_1,w_2) + \\
&\sum_{l=1}^{2}\mathcal{F}^{-1}_{(\xi_1\xi_2)}\left[\frac{d_1d_2i\xi_lk_{21}k_{22}(k_{22}+\omega_\lambda)}{\mathfrak{m}_2(\xi_1,\xi_2,\lambda)}\mathcal{M}_0(w_3)\hat{h}_l(0)\right](w_1,w_2). \\
&= \mathfrak{A}^2(\lambda)\mathcal{G}_\lambda\boldsymbol{G}
\end{aligned} \tag{40}$$

$$\begin{aligned}
u_j \quad &= \mathcal{F}^{-1}_{(\xi_1,\xi_2)}\left[\hat{l}_j(0)e^{-\omega_\lambda w_3}\right](w_1,w_2) - \\
&\sum_{i=1}^{2}\mathcal{F}^{-1}_{(\xi_1,\xi_2)}\left[\frac{i\xi_j(k_i+\omega_\lambda)L_{i1}}{\mathfrak{m}_i(\xi_1,\xi_2,\lambda)}\mathcal{M}_i(w_3)\hat{g}(0)\right](w_1,w_2) + \\
&\sum_{i=1}^{2}\sum_{l=1}^{2}\mathcal{F}^{-1}_{(\xi_1,\xi_2)}\left[\left(\frac{(-1)^id_1d_2\xi_j\xi_lk_{21}k_{22}(k_{22}+\omega_\lambda)}{d_i\mathfrak{m}_i(\xi_1,\xi_2,\lambda)}\mathcal{M}_i(w_3)\right)\hat{h}_l(0)\right](w_1,w_2) \\
&:= \mathfrak{B}_j^2(\lambda)\mathcal{G}_\lambda\boldsymbol{G}, \text{ dengan } j = 1,2.
\end{aligned} \tag{41}$$

$$\begin{aligned}
u_3 \quad &= \sum_{i=1}^{2}\mathcal{F}^{-1}_{(\xi_1\xi_2)}\left[\frac{k_i(k_i+\omega_\lambda)L_{i1}}{\mathfrak{m}_i(\xi_1,\xi_2,\lambda)}\mathcal{M}_i(w_3)\hat{g}(0)\right](w_1,w_2) + \\
&\sum_{i=1}^{2}\sum_{l=1}^{2}\mathcal{F}^{-1}_{(\xi_1\xi_2)}\left[\frac{(-1)^id_1d_2i\xi_lk_{21}k_{22}k_i(k_i+\omega_\lambda)}{d_i\mathfrak{m}_i(\xi_1,\xi_2,\lambda)}\mathcal{M}_i(w_3)\hat{h}_l(0)\right](w_1,w_2) \\
&:= \mathfrak{B}_3^2(\lambda)\mathcal{G}_\lambda\boldsymbol{G}.
\end{aligned} \tag{42}$$

Let $\mathfrak{B}^2(\lambda) = \left(\mathfrak{B}_1^2(\lambda), \mathfrak{B}_2^2(\lambda), \mathfrak{B}_3^2(\lambda)\right)^{\mathsf{T}}$. Then the solution operator $\mathbf{u}$ can be expressed as:

$$\mathbf{u} =: \mathfrak{B}^2(\lambda)\mathcal{G}_\lambda\boldsymbol{G}. \tag{43}$$

Thus, the solution operator $(\rho, \mathbf{u})$ can be written as:

$$(\rho, \mathbf{u}) = (\mathfrak{A}^2(\lambda)\mathcal{G}_\lambda\boldsymbol{G}, \mathfrak{B}^2(\lambda)\mathcal{G}_\lambda\boldsymbol{G}). \tag{44}$$

Therefore, the solution operators $(\rho, \mathbf{u}) = (\mathfrak{A}^2(\lambda)\mathcal{G}_\lambda\boldsymbol{G}, \mathfrak{B}^2(\lambda)\mathcal{G}_\lambda\boldsymbol{G})$ for the equation (16) in the half-space $(\mathbf{R}_+^3)$ is obtained. Hence, Theorem 3 tis proven.

Next, proving Theorem 2. Reconsider equation (14) the solution operator in the half-space $(\mathbf{R}_+^3)$ is:

$$\begin{aligned}
\rho \quad &= M + \tilde{\rho} \\
&= \mathfrak{A}^1(\lambda)\mathcal{K}_\lambda^0\mathbf{K}^0 \ + \mathfrak{A}^2(\lambda)\mathcal{G}_\lambda\boldsymbol{G} \\
&= \mathfrak{A}^0(\lambda)\mathcal{K}_\lambda^0\mathbf{K}^1
\end{aligned}$$

Next, if $\mathbf{u} = \mathbf{P} + \tilde{\mathbf{u}}$, then $\mathbf{u}$ can be written as:

$$\begin{aligned} \mathbf{u} &= \mathbf{P} + \tilde{\mathbf{u}} \\ &= \mathfrak{B}^1(\lambda)\mathcal{K}_\lambda^0\mathbf{K}^0 + \mathfrak{B}^2(\lambda)\mathcal{G}_\lambda \boldsymbol{G} \\ &= \mathfrak{B}^0(\lambda)\mathcal{K}_\lambda^0\mathbf{K}^1 \end{aligned}$$

It can be conclude that the system of equations (3) for the two cases of coefficient, namely $\left(\frac{\mu_*+\nu_*}{2\kappa_*}\right)^2 - \left(\frac{1}{\kappa_*}\right) < 0$ and $\left(\frac{\mu_*+\nu_*}{2\kappa_*}\right)^2 - \left(\frac{1}{\kappa_*}\right) > 0, \kappa_* \neq \mu_*\nu_*$, has the solution operator in the half-space ($\mathbf{R}_+^3$) given by $(\rho, \mathbf{u}) = \left(\mathfrak{A}^0(\lambda)\mathcal{K}_\lambda^0\mathbf{K}^1, \mathfrak{B}^0(\lambda)\mathcal{K}_\lambda^0\mathbf{K}^1\right)$. Thus, Theorem 2 is proven.

CONCLUSIONS AND RECOMMENDATIONS

In this research, it can be conclude that Theorem2 is proven to have the solution operator for the Navier Stokes Korteweg model with Neumann boundary conditions in the half-space ($\mathbf{R}_+^3$). The solution operator for the two cases of coefficients $\left(\frac{\mu_*+\nu_*}{2\kappa_*}\right)^2 - \left(\frac{1}{\kappa_*}\right) < 0$ and $\left(\frac{\mu_*+\nu_*}{2\kappa_*}\right)^2 - \left(\frac{1}{\kappa_*}\right) > 0, \kappa_* \neq \mu_*\nu_*$ is given by $(\rho, \mathbf{u}) = \left(\mathfrak{A}^0(\lambda)\mathcal{K}_\lambda^0\mathbf{K}^1, \mathfrak{B}^0(\lambda)\mathcal{K}_\lambda^0\mathbf{K}^1\right)$ for the system of equations (3). Future research is expected to estimate the solution operator by examining R-bounded cases and solving the resolvent system in the bent half-space (Ω_+).

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